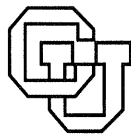


ARITHMETIC IN SCHOOLS, PAST, PRESENT, AND FUTURE

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Arithmetic in Schools, Past, Present, and Future

Abstract

This paper discusses the content of arithmetic taught in American schools from 1780 to the present, and proposes some radical changes.

Arithmetic in Schools, Past, Present, and Future

Old books in arithmetic do not make much sense from the point of view of modern mathematics, but they do make sense from the point of view of computer science. Students of arithmetic were not learning mathematics, but they were trained to be human computers. They were learning to perform efficiently work that can now be fully automated. Mathematics and computer science deal with different aspects of arithmetic. Mathematical properties of integers, real and rational numbers, and relations among them are summarized in definitions, axioms, and theorems. Computer science deals not with numbers but with data structures, which are representations of numbers, and with processes of computation that are described by numerical algorithms. The textbooks in arithmetic printed in the United States from 1780 until the present day closely correspond to a computer science view of arithmetic and completely ignore its mathematical aspects.

Summary of the content of books of arithmetic

In the early period (1780 - 1850), one textbook usually covered the whole topic of arithmetic (e.g., Pike, 1806; Adams, 1808; Daboll, 1823; Ostrander, 1823); and after 1825 there were special books for teaching young children (Emerson, 1829; Colburn, 1826). After 1850, several series of books at different levels of difficulty were published (e.g., Fish 1858-1877; Ray, 1877; 1881), the highest level containing the whole arithmetic, and lower levels adapted to younger learners (e.g., French, 1871). Around 1885 graded books were introduced (Harper, 1883; Prince, 1894-1902; Nichols, 1898-99), and they become standard after 1900 (Wentworth-Smith, 1919, 1920; Thorndike, 1917; and many others). Each book was intended for one grade. Therefore, in spite of their huge overlap, no single book contained the whole material. The role of textbooks also changed during this period. Early texts were meant mainly for teachers; they were "schoolmasters' assistants" (e.g., Dilworth, 1762). Later the same text was used by both the teacher and the students. At present we often have two separate books for each grade, one for the teacher and the other for students, with additional booklets with enrichment materials and worksheets for practice.

Content of a generic early book (circa 1830) about arithmetic

The abstract part of arithmetic consists of studying numbers and processes. There were three types of numbers in early books:

Unsigned integers, in three different notations:

Hindu-Arabic notation (examples: 12, 105),

Roman notation (examples: XII, CV), and

in (English) words (examples: twelve, one hundred five).

Fractions and mixed numbers, subdivided into two categories:

common (also called vulgar) fractions (example: $2/3$),
decimals (example: 23.1).

Surds, which were selected irrational numbers (example ; $\sqrt{2}$).

There were six basic processes:

numeration, which was the skill of reading numbers,
notation, which was the skill of writing numbers,
addition,
subtraction,
multiplication, and
division.

(Notice that the processes of reading and writing are *not* mathematical operations on numbers, but they *are* programming constructs in computer science.)

Two additional processes were:

involution, which was computing a power with an integral exponent, and
evolution, which was root extraction, limited to square and cubic roots.

For each operation there were several algorithms, depending on the data types involved. So for addition there were separate algorithms for:

addition of integers,
addition of common fractions and mixed numbers, and
addition of decimals.

Applications often took more than half of the whole text. This topic extended the basic processes to

denominate numbers (examples: 3 ft.; 5 lb.; \$5.20), and
composite numbers (examples: 5 ft., 3 in.; or 1 gross, 3 dozen, 1),
and added algorithms appropriate for these data structures.

Operations on denominate numbers were severely restricted. Here are examples of restrictions:

"You may only add numbers of the same kind."

"You have two types of division. In the first one, both the dividend and the divisor are of the same denomination, and the quotient is abstract. In the second one, the divisor is abstract and the quotient has the same denomination as the dividend."

"In multiplication, the first factor is abstract and the second one is concrete."

Two additional processes were introduced:

computing percentage, and
solving proportions.

Arithmetic was applied to two main domains: mensuration (measuring) and finances. General (algebraic) formulas were never given. Each application was presented as a numerical example.

Often specific professional groups were directly targeted:

merchants, farmers (Daboll, 1823), masons, carpenters, lumber men (Hawney, 1813), and so on. (At that time, artisans commonly used a slide rule, and merchants used arithmetic tables, sometimes called "Ready Reckoners," but arithmetic books dealt only with paper and pencil calculations. (Day, 1866; The World's Ready Reckoner and Rapid Calculator (no author given), 1890)

Every arithmetic book contained many exercises for practice. There were two main types of exercises: for practicing processes (at present these are called worksheets) and word problems. The first were for training skills in computation, and the second were for learning which situation calls for which process.

Is school arithmetic a part of mathematics?

It is easy to recognize that early books about arithmetic contain many concepts extensively studied within several mathematical theories. The positive integers (natural numbers) are the subject of study of number theory, which has been and is a hobby of many mathematicians (Hardy, 1940). Fields of rational and real numbers are the subject of study of both algebra and calculus. Algorithms for addition, subtraction, multiplication and division were studied in numerical analysis, and in the relatively new theory of algorithms.

- But in books of arithmetic no theorems are quoted, and the concept of proof, when it is used, means only to check for errors.

For example if you are adding a list of numbers, you "prove" correctness by adding the numbers again, preferably in a different order. It is clear that the aim of arithmetic was to train students in the hand execution of a few specific numerical procedures and not to teach them about general properties of numbers, or to introduce them to the general concept of an algorithm. In addition the jargon of arithmetic did not include variables, which made even the formulation of any general statement rather difficult.

- Was then school arithmetic a part of mathematics? Traditionally it is classified as such. But if we judge by the content, the answer is no.

School arithmetic did not contain any theorems or proofs. It did not include any deductive reasoning. Looking back to ancient times, in Greece there were two concepts: arithmetic, and logistics (D. E. Smith, 1923a,1951; 1925, 1953; Karpinski, 1965; Boyer, 1989). The first one, arithmetic, corresponding to modern number theory, was the science of numbers, which was an important part of mathematics. The second, logistics, corresponding to modern school arithmetic, dealt with skills in computation, and was not considered to be a part of mathematics (Brooks, 1890).

What has remained the same between 1780 and the present time

When we look at the content of arithmetic in modern textbooks for grades one to eight, we can recognize most of the elements shown above, with some omissions and few additions. So the content remains the same. The main differences we see between old and new books are not in the arithmetic that is taught, but in two other aspects of teaching and learning, namely goals and methods, and in the topics included in applications.

Changing goals in teaching arithmetic

We can identify different goals in teaching arithmetic that correspond to three different time periods. (The dates are approximate and there are no sharp boundaries.)

Before 1820. Arithmetic was part of vocational education, and a student was viewed as an apprentice of an artisan, a farmer, or a merchant.

Goal. Student becomes a proficient computer who expertly applies his skills in his profession.

1820 - 1960. Arithmetic was a part of general education, and a student was viewed as a human computer and a member of society (McLellan & Dewey, 1895).

Goal 1. Student becomes a good computer who can apply his or her skills to practical problems.

Goal 2. Student improves his or her ability to think logically and develops good working habits.

1960 - present. Arithmetic is still a part of general education, and a student is viewed as a little mathematician and a developing human (Bruner, 1963).

Goal 1. Student becomes a good problem solver who can use his or her problem solving ability in everyday life.

Goal 2. Student improves his/her higher thinking skills, and his/her self confidence.

(Notice that teaching arithmetic as a part of general education also has developmental goals. These goals are based on a common belief in the intrinsic educational value of arithmetic (M. Smith, 1994; D.E. Smith, 1900, 1923b; NCTM, 1989).

Teaching methods

The earliest books employed the "synthetic" method of teaching. Students first learned numeration and notation. Next they memorized "facts," and learned processes on a few very complex examples. Finally they practiced for speed and accuracy and did word problems to learn when to apply which process. This method was not suitable for young children, so another method, called "analytic," "inductive," or "progressive," first used by the Swiss educator Johann

Pestalozzi (1746-1827), was introduced (see Adams, 1845). This method, which underlies the modern spiral curriculum, spread rather slowly but became a standard for graded texts. Students start with simple examples and work with objects and pictures, slowly progressing to more complex cases. They practice all the time and solve word problems even before they learn notation and numeration. The methods of Pestalozzi have brought a heavy use of manipulatives in the early grades. But the use of pictures and illustrations has increased very slowly to the current level.

Changes in content before 1960

The topics included in applications of arithmetic have been changing all the time, reflecting changes in technology, business practices, and social standards. The types of exercises have remained the same. The main ones were practicing computation and doing word problems. The content of word problems changed together with changes in applications, and the complexity of exercises reflected current pedagogical thinking about the level of difficulty appropriate for a specific grade, and the targeted levels of skill.

There has been one additional change. Early books did not include counting as one of the processes. It was probably assumed that older children or young adults already know how to count without any formal schooling. Later, counting became the first process that children practice.

New Math

Around 1960 a radical attempt to "modernize" school mathematics started. Within higher mathematics two theories played crucial roles, providing a foundation for other theories and unifying different ideas. They were set theory and "modern" algebra. Without realizing that school arithmetic is not a part of mathematics, but is the training of some specific procedural skills, some mathematicians and educators attempted to embed it in set theory, and embellish it with algebraic jargon. So if John and Mary put their marbles together they were forming a union of two disjoint sets. Simple equality, $2 + 1 = 1 + 2$, was an instance of commutativity of addition. Even mathematical logic got its share in this madness when computation became "renaming the number." Mathematical jargon is only meaningful within a specific mathematical theory. Taken out of context, it either becomes meaningless or takes on another meaning. (For example, in the jargon of arithmetic the phrase, "the sum of ten ones," became synonymous with "the set of ten ones.") The attempt to give a mathematical foundation to school arithmetic was soon abandoned, leaving behind some small changes in the jargon of arithmetic and a strong distrust of any radical change in the status quo (Kline, 1973; Conference Board of the Mathematical Sciences, 1975).

The present

The availability of personal computers and calculators has made paper and pencil (but not mental) calculations obsolete. So skills in executing arithmetic algorithms on paper will be completely useless when present day children reach adulthood. This is recognized by many educators, and by the National Council of Teachers of Mathematics (NCTM. 1989).

The main recommendation presented by the NCTM is:

Teach for understanding, not for specific skills.

And the main goal is:

Children should become good problem solvers.

In 1989 the NCTM recommended gradual changes in both content and methods of school mathematics, and since then there has been a plethora of suggestions and proposals. Some deal with enrichment materials, such as exercises in guessing a pattern, or collecting statistical data. Others deal with methods of teaching and classroom management, "Children cannot be taught; they have to construct their own knowledge" (Steffe, Cobb, & von Glaserfeld, 1988), or "The teacher must be a facilitator and a guide" (Bosworth & Hamilton, 1994). All these good attempts break down at the classroom level, because the core of the elementary school "math" curriculum is arithmetic, which is really not a part of mathematics (no theorems, no reasoning) but is instead training in the execution of specific algorithms.

- Thus, within the framework of present day school arithmetic, opportunities to teach for understanding are very limited.

This situation is not satisfactory. The National Educational Assessment Program (NEAP) tests show that most students going through a "math" program get out without skills and with almost no understanding of mathematics. They have very little to show after thousands of hours of schooling.

The future

Because learning paper and pencil calculations is acquiring an obsolete skill, we may expect school arithmetic to disappear slowly from the curriculum. The question is, what will take its place? At present there is no comprehensive proposal under discussion, and the existing trend is to clutter the curriculum with disjoint fragments of different parts of mathematics, taken out of context. Also we see some attempts to find a "kingly road" to mathematics. For example, the very insightful remarks by Polya (1957) about problem solving, based on teaching Euclidean geometry, are taken as a general strategy for solving any problem, independent of the knowledge specific to a particular domain (e.g., Bennett & Nelson, 1992).

A proposal

Content.

We propose to:

1. Restrict the training of skills to mental computations and use of different computing devices (calculators and computers).
2. Replace school arithmetic by mathematics by teaching as one unit:
 - Theory of real numbers;
 - Number theory;
 - Numerical algorithms.
3. Provide a broad range of applications to problems that are interesting for children of elementary school age.

Such a change in elementary schools could lead to additional changes in middle school and high school.

1. One could teach as one topic: algebra, analytic geometry and calculus, with the support of such software tools as Mathematica.
2. One could teach discrete mathematics and programming on the basis of one or more major programming languages.

We think that the proposed mathematics in elementary school should be for everyone. But in high school, students should be able to choose instead a course in the use of technology that teaches survival skills in a high-tech world for those who do not want to learn either mathematics or programming (M. Smith, 1994).

Methods

Such changes in content would require changes in the method of teaching. Both worksheets and word problems, which are part of the training of skills, and have no other intellectual value, could be abandoned and replaced by project-type problems (Baggett & Ehrenfeucht, 1995a). This parallels at the school level the changes in the teaching of undergraduate calculus (King, 1992; Cohen, Gaughan, Knoebel, Kurtz, & Pengelley, 1991; etc.).

Are changes possible?

Any large scale change in the content and methods of teaching would require several decades to implement. What is taught and how it is taught depends on teachers, and at any given moment there are very few teachers who are willing to change their methods of teaching or to experiment with new content. The same is true about the teachers of teachers. Schools of education, which train future teachers, are very conservative, as can be seen by comparing new and old books used in math education courses. Thus, any successful change has to start small. If some teachers, schools, and schools of education adopt a new approach, and if the results are visibly better and closer to accepted

educational goals, then the changes will spread and slowly become a new standard.

Our proposals for middle and high schools are not radical. The problems of unifying school mathematics, making it relevant, using computer technology, and modifying requirements depending on children's interests and career goals (M. Smith, 1994; Gardner, 1983) are broadly discussed, and there is hope that some consensus will slowly emerge.

Our proposal for elementary school is radical. Mathematics is abstract. Applications of mathematics, also called mathematical modeling, consist of matching abstract mathematical concepts with concrete concepts of the physical world or with concepts from the existing economic and social environment. Thus learning mathematics is learning abstract thinking, and not practicing procedures. Most of the existing psychological theories of children's learning agree with the Piagetian contention that children in the age group that attends elementary school are developmentally unable to think abstractly. This leads to a belief that strongly influences the early teaching of mathematics. (This belief is also in agreement with the Montessori method (Montessori, 1912).)

- A statement of the belief: In early grades children can only learn concrete concepts, and this prepares them for learning abstract concepts in the future. Thus there can be no real learning of mathematics in early grades.

Can young children think abstractly?

To be more specific, can young children state, understand, and formulate mathematical theorems? Can they justify them by deductive reasoning? Can they perceive the application of mathematics as an approximate matching between abstract and concrete?

- Disagreeing with the Piagetian legacy, we answer positively to all questions above.

Young children learning mathematics for approximately one school period per week (Baggett & Ehrenfeucht, 1995a) show a considerable level of mathematical sophistication (Baggett & Ehrenfeucht, 1994, 1995b). Below we show by examples the level of understanding and reasoning that can be rather easily achieved by most children in first and second grades.

First grade. (Children have been using calculators for all more complex calculations.) "There is no biggest number." (Why?) "You can always add one and get a bigger number."

Second grade. ("Number" means decimal number. The distinction between rational and irrational numbers is still unknown. Children use all four arithmetic

operations.) "Between any two numbers there is another number." (How do you know?) "You add them, divide by 2, and you have a number in the middle."

Some general observations.

- Young children (first through third grades) are interested in numbers. When we asked them what they like about mathematics, the most common answer was, "Numbers."
- Children have more difficulty in going from abstract to concrete than from concrete to abstract. For example, drawing a simple polygon (e. g. a rectangle) with a given perimeter is more difficult than measuring and computing the perimeter of a fairly complex polygon (e. g. a non-convex hexagon).

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