Subtraction Strategies of Second and Fourth Grade Children on Problems Containing Whole Numbers, Decimals, and Negative Numbers

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Abstract

This study investigates strategies of children who are doing difficult subtraction problems that involve both decimals and negative answers, when calculators are available. Eighty-four of the 94 second and fourth grade children who were tested showed consistent strategies, and they were successful in using their strategies: 86% of their answers were correct relative to the strategies they chose. But 53% of these children chose an *incorrect* strategy which systematically gave an incorrect answer in some situations. Implications for the incremental method of teaching, and for the role of practice, are discussed.

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There is no standard curriculum in elementary mathematics in the early grades. But different curricula that have been adopted by different states, school districts, and schools have many things in common. Two common aspects of many curricula, and of many textbooks, are that there are 'strands', such as whole numbers, geometry, and measurement; and that the instruction starts with easy problems and gradually progresses to problems of greater difficulty. (Modern terms for this sequencing include 'spiral' or 'incremental' teaching, or 'learning in stages'.) As an example, Appendix 1 shows a page from the Framework for the Michigan Mathematics Objectives (Michigan State Board of Education, 1989), which incorporates 'strands'.

The Michigan Framework (in Michigan State Board of Education, 1989) suggests that teaching be incremental. For example, under "whole numbers" is the following (p. 16):

"Subtraction: Three-digit, one regrouping in grades K-3; three-digit, two regroupings in grades 4-6. (This objective should be mastered by the end of the fourth grade.)"

Under Decimals (p. 76) is the following:

"4-6 and 7-9 Comment:
Paper and pencil computation parallels the limits used with whole numbers. Calculators are used for larger multi-digit computations. Limit grades 4-6 to whole numbers, tenths, and hundredths."

Textbooks typically attempt to mirror such curricular frameworks, both in the materials they contain, and in the order in which the materials are presented. Flanders' (1994) research indicates that teachers rely highly on their math textbooks for topic sequencing. So it is common in actual practice that subtraction is taught in stages, as are many other mathematical topics.

Arranging mathematics instruction for young children in stages, so that it begins with simple examples and proceeds to more complex ones, was introduced in the United States by Warren Colburn (1793-1833; see his books of, e.g., 1826 and 1849) and by Frederick Emerson (1788-1857; see his 1829 book). This sequencing, which historically was termed 'analytic' or 'inductive', (Adams, 1848), or 'progressive' (Fish, 1867; 1868), had its roots with Johann Pestalozzi (1746-1827) in Switzerland. It can be contrasted with the 'synthetic' method, in which one proceeds from a general rule to examples. In the preface to his second arithmetic book, dated 1848, Daniel Adams (1773-1864), explains this difference, alluding to his first book, Scholar's Arithmetic (1801):

"There are two methods of teaching, -- the *synthetic* and the *analytic*. In the *synthetic* method, the pupil is first presented with a *general* view of the science he is studying, and afterwards with the particulars of which it consists. The analytic method *reverses* this order: the pupil is first presented with the *particulars*, from which he is led, by certain natural and easy gradations, to those views which are more general and comprehensive."

"The Scholar's Arithmetic, published in 1801, is synthetic. If that is a *fault* of the work, it is a fault of the *times* in which it appeared. The analytic or inductive method of teaching, as now applied to elementary instruction, is among the improvements of later years. Its introduction is ascribed to Pestalozzi, a distinguished teacher in Switzerland. It has been applied to arithmetic, with great ingenuity, by Mr. Colburn, in our own country."

After Colburn and Emerson introduced the analytic method, it slowly took over in American schools, and, as has been noted, it is still found today in the teaching of subtraction. In the earliest grades (kindergarten and first grade) children are taught the concept of 'take away' with manipulatives and pictures. Then in first and second grade they write the addition and subtraction facts. Subtraction of two-digit whole numbers, first without regrouping, and next with regrouping, occurs in second grade. Then comes subtraction of three-digit numbers, first with one regrouping and then with two (as in the Michigan Framework). Then comes subtraction of decimals, and finally, usually in algebra, children learn about negative numbers. Subtraction of common fractions is a separate topic.

This plan of teaching has not changed since Colburn and Emerson, even though calculators have become widely available in the early grades, giving one the opportunity to abandon such sequencing. (In 1989 the NCTM recommended the use of calculators, in all grades, in its Curriculum and Evaluation Standards for School Mathematics.) Also some research studies (e.g., Carpenter, Ansell, Franke, & Fennema, 1993) indicate that children can solve a wide range of problems much earlier than generally has been presumed, suggesting that such sequencing may be unnecessary. Further, a number of researchers (Fuson, 1990 ; Graeber & Tirosh, 1990; Maurer, 1987) have questioned the wisdom of incremental sequencing.

Methodology

The question we examine in this paper is how second and fourth grade children perform subtraction when calculators are available. We wanted to see how children handle subtraction problems which they have not seen before, and which include more advanced concepts such as decimals and negative numbers (see Table 1). The problems were given to children by their classroom teacher in the familiar setting of doing worksheets. All children (with one exception) promptly started working on them. The teachers did not report any surprise or frustration on the part of the children due to the rather unusual content of the worksheets. Looking at the solutions the children provided gave us better insight about their concepts of subtraction than we could get from looking at subtraction of two-digit whole numbers, which is more standard for second grade.

There is much research about errors in whole number subtraction (e.g., Buswell (1926), Cox (1975), Brown & vanLehn (1980), vanLehn (1982, 1986), Ashcroft (1994)). But to our knowledge this is the first study to examine subtraction strategies involving decimals and answers having negative numbers, giving it a broader scope than previous studies.

Subjects.

Ninety-four children in four classrooms (three second grades and one fourth grade) in two schools in the Midwest participated in the study. All three second grade classrooms were from the same school, and constituted the entire second grade population at the school. The fourth grade class was from a different school in the same school district. Both schools were in small rural towns. There was no tracking in either school or in any classroom; children of varying abilities were present in all classrooms. The children had been using calculators in their mathematics lessons approximately once a week for the entire school year, but they had not done any worksheets involving decimals or negative numbers. Their teachers had attended workshops on calculator use, and were given classroom materials for problem solving with calculators.

Procedure.

Late in the spring semester the classroom teachers gave each of their children a copy of a printed subtraction test consisting of nine problems. The test is shown in Table 1. It was given in the regular classroom setting, and although the children had not had training or practice in these types of problems, the worksheet format was familiar to them. Each child also had a pencil and a calculator. Children were not required to use the calculators on the test. They were asked to work alone, to use their calculators if they wanted to, and to record their answers on the sheets. Teachers were asked not to help the children.

Subtract:

747 82	8565 2402	56308 8456
1334 1901	5693 5693	184 1467
57.7 6.54	5.58 5.32	7.11 10.23

Table 1. Subtraction test given to children.
Answers are given in Table 3.

Results and Discussion

1. Categories of answers.

Of the ninety-four children, one child turned in a test with all answers blank. Most of the remaining ninety-three children gave answers to all problems. We divided them into groups, depending on their knowledge of negative numbers and decimals. There were four possible categories -- children who demonstrated a knowledge of (a) both negative numbers and decimals, (b) negative numbers and not decimals, (c) decimals and not negative numbers, and (d) neither negatives nor decimals. The criteria we used for deciding on the knowledge they had were whether they used a minus sign in at least one answer, and whether they used a decimal point in at least one answer. This does not mean that children in the first group understand decimals and negative numbers. But it shows that they consider that a decimal point and a minus sign are parts of the numerals that they write.

We know that some children consider that what is after the decimal point is cents -- they voluntarily write 'cents' for that part. Some children ignore the minus sign. In some cases children interpret the minus sign as an error message -- "you get a wrong answer," or "you shouldn't do it." The categories above reflect different children's concepts of a numeral. Table 2 shows the number of children in each of the four groups. (Notice that no children use a minus sign but do not use a decimal point.)

Table 2. Number of children (out of 93) who demonstrated a knowledge of negative numbers, of decimals, of both, or of neither

		negative numbers	
		YES	NO
decimals	YES	50(1)	27(2)
	NO	0	16(3)

Notes:
There were no children who demonstrated a knowledge of negative numbers but did not demonstrate a knowledge of decimals.
The three groups are numbered above as (1), (2), and (3).

2. Strategies.

We found that most children used some global strategies (algorithms) consistently. Consistency means that they solved all problem using the same strategy. Global means that the strategy provided an answer in all cases.

In group 1 (children who know about both decimals and negative numbers) there were two strategies:

- Correct: (39 of the 50 children in group 1 showed this strategy).
- Top-from-bottom: subtract the top number from the bottom one (7 children showed this strategy). Here is an example of the top-from-bottom strategy:

$$\begin{array}{r} 747 \\ \underline{82} \\ -665 \end{array}$$

In examining the set of answers for each of the remaining four children in group 1, we observed that two of them switched from the top-from-bottom strategy to the correct one. These children were not consistent. Half way through the test they changed their strategy. We did not detect any strategy in the answers of the remaining two children.

Group 2 (children who knew decimals, but did not know negative numbers), with 27 children, displayed only one strategy:

• Smaller-from-larger: subtract the smaller number from the larger one (26 children showed this strategy). Here is an example of the smaller-from-larger strategy:

$$\begin{array}{r} 184 \\ \underline{1467} \\ 1283 \end{array}$$

We did not detect any pattern in the answers of one child in group 2.

Group 3 (children who did not know decimals and did not know negative numbers) consisted of 16 children, and it provided the largest variety of strategies:

• Leave problems with decimals blank: four children did not give any answers to problems that contain a decimal point. We classify these children as not having a global strategy because their strategy did not handle numbers with a decimal point.

• Ignore: ignore the decimal point, and subtract the smaller number from the larger one (six children showed this strategy). Here is an example of the ignore strategy (note that the alignment of digits is changed):

$$\text{given:} \quad \begin{array}{r} 57.7 \\ \underline{6.54} \\ \end{array} \qquad \text{child's reading:} \quad \begin{array}{r} 577 \\ \underline{654} \\ 77 \end{array} \quad (= 654 - 577)$$

• Digit: subtract the smaller digit from the larger one (five children showed this strategy). Here is an example (note that digit alignment is not changed):

$$\begin{array}{r} 57.7 \\ \underline{6.54} \\ 51.24 \end{array}$$

We note that this error, sometimes called a 'reversal' in the literature (Carpenter, 1975; Buswell, 1926), is one of the most frequently occurring errors in hand computation involving subtraction of multi-digit numbers.

Finally, one child (out of 16 in this group) showed no recognizable strategy.

3. Answers correct relative to a strategy.

Table 3 shows which answers are correct relatively to each strategy. Table 4 shows how well a child's answers matched his or her strategy. Namely, it shows the percentage of correct answers, relative to each of the strategies. The table indicates that for each of the five strategies we were able to identify, children used the strategy consistently for about 86% of their answers.

Table 3

Answers to subtraction problems according to each of the five strategies:

	747 82	8565 2402	56308 84561	1334 1901	5693 5693	184 1467	57.7 6.54	5.58 5.32	7.11 10.23
Strategy:									
Group 1 correct	665	6163	47852	-567	0	-1283	51.16	0.26	-3.12
Group 1 top-from-bottom	-665	-6163	-47852	567	0	1283	-51.16	-0.26	3.12
Group 2 smaller-from-larger	665	6163	47852	567	0	1283	51.16	0.26	3.12
Group 3 ignore	665	6163	47852	567	0	1283	77	26	312
Group 3 digit	745	6163	52152	633	0	1323	5124	26	1712

Notes.
Group 1 demonstrated a knowledge of both decimals and negative numbers.
Group 2 demonstrated a knowledge of decimals but not negative numbers.
Group 3 demonstrated a knowledge of neither decimals nor negative numbers.

Table 4. Percentage correct relative to a strategy (number of children using a given strategy is shown in parentheses)

Group	strategy	% correct
1	Correct (39)	88%
	Top-from-bottom (7)	94%
2	Smaller-from-larger (26)	86%
3	Ignore (6)	81%
	Digit (5)	71%
	Overall percentage correct relative to a strategy	86%

Table 5 shows the distribution of the number of correct answers that children recorded (relative to their strategies). We note that all children got at least four correct answers relative to their strategy. This may be an artifact of the data analysis. If a child made too many errors relative to his or her strategy, we would not be able to identify the strategy. It is possible that two children in groups 2 and three we were not able to classify simply made too many errors relative to their strategies.

Table 5. Distribution of the number of correct answers (relative to strategies)

Number of correct answers	% of students (n = 83)
9	39%
8	27%
7	14%
6	16%
5	1%
4	4%
less than 4	0%

4. Differences between different classrooms.

Table 6 shows the distribution of the five strategies among the four different classrooms. One of the second grade classrooms shows a noticeably poorer performance than the other two. And the fourth grade children do not perform better than the second grade children. Seven children in fourth grade make the error 'top-from-bottom', and they are the only ones who make this error. We have noted such class-specific differences before (e.g., Baggett & Ehrenfeucht, 1994), and they are mostly due to differences between classroom teachers.

Table 6. Distribution of strategies among classrooms (number of children)

	classroom: 4th grade	2nd grade	2nd grade	2nd grade
number of children in class:	(27)	(23)	(21)	(23)
strategy:				
correct	10	18	11	0
top-from-bottom	7	0	0	0
smaller-from-larger	8	2	9	7
ignore	0	0	1	5
digit	1	0	0	4
No global strategy	1	3	0	7

(10 children showed no strategy; 1 had a blank sheet)

Summary

- 83 out of 94 children formed a global subtraction strategy that always produced a result, *within their concept of a numeral.* As discussed above and shown in Table 2, we found that a child's concept of number might include decimals and negative numbers or decimals (but not negatives) or neither. (We found no case of a child who used a minus sign but did not use a decimal point.)

- 44 of the 83 children who demonstrated a global strategy (53%) formed an *incorrect* strategy (see Table 3).

- Strategies were used successfully: 86% of answers were correct, relative to a child's strategy (Table 3); and children who used a strategy used it successfully in at least four of the nine problems (Table 4).

- There were large differences in the distribution of strategies between classrooms and within one classroom. (See, as we have noted, the poor performance of the second graders in the classroom shown in the rightmost column of Table 5.) Fourth graders did not outperform second graders.

Comments

The incremental method of teaching.

The belief behind the incremental method of teaching (also called teaching in stages, analytic, inductive, or progressive teaching) is that over time and with new instruction children will expand their concept of numbers and extend their previously learned algorithms to cover new cases. The results of this study suggest, however, that children form global strategies which they spontaneously use on new cases. In over fifty percent of the cases, these strategies are incorrect, and these children therefore will have to *unlearn* previously learned strategies in order to learn new ones. (In previous research, Englehardt & Usnick (1991) suggested that more attention be given to teaching methods that begin with more general cases. Fuson (1990) also proposed that in multi-digit subtraction, problems with and without "trades" (i.e., regrouping) be presented at the same time, and that "All possible combinations of trades are done from the beginning" (p. 274). Other researchers, e.g., Graeber et al (1990) and Maurer (1989) have made similar suggestions.)

On the role of practice.

The results of this study also call into question the value of extensive practice. The children who were tested here showed a high rate of success (over 86%) in applying their strategies, and we know that they had not practiced on problems with negative numbers and decimals before. Therefore one can conclude that the poor performance of children on all arithmetic tasks in general is not because they lack practice, but because they use incorrect strategies.

Suggestions for changes in instruction.

There is a simple solution for teaching general strategies for subtraction, strategies that do not have to be unlearned. We suggest that in teaching subtraction, one start by showing children that a numeral may have three characteristics: a sign, some digits, and a decimal point between some digits.

Before the decimal point is the whole part, and after the decimal point is a decimal fraction. (A sign and a decimal point do not always occur.) Then children can do subtraction both mentally and with a calculator, and they can be shown when a minus sign occurs and when a decimal point occurs. We think that with this instruction, which bypasses the 'synthetic' method (Adams, 1808) and which clearly breaks from the analytic method of Colburn, Emerson, and Pestalozzi, the teaching of correct strategies for subtraction could be completed by the third grade.

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Appendix

1. From Michigan State Board of Education. (1989) An Interpretation of: The Michigan Essential Goals and Objectives for Mathematics Education. Lansing, MI: Michigan Council of Teachers of Mathematics, p. vii.

Framework:
Michigan Mathematics Objectives

Mathematical Processes

Mathematical Content Strands

	Conceptualization	Mental Arithmetic	Estimation	Computation	Applications and Problem Solving	Calculators and Computers
Whole Numbers and Numeration						
Fractions, Decimals Ratio and Percent						
Measurement						
Geometry						
Statistics and Probability						
Algebraic Ideas						
Problem Solving & Logical Reasoning						
Calculators						