A Non-nested Coarse Space for Schwarz Type Domain Decomposition Methods

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Abstract

In this paper, we develop a new technique, and a corresponding theory, for Schwarz type overlapping domain decomposition methods for solving large sparse linear systems which arise from finite element discretization of elliptic partial differential equations. The theory provides an optimal convergence of an additive Schwarz algorithm that is constructed with a non-nested coarse space, and a not necessarily shape regular subdomain partitioning. It allows the use of subdomains with non-uniform aspect ratio and non-smooth boundaries. The theory is also applicable to the overlapping graph partitioning algorithms recently developed by Cai and Saad [5], and to the non-nested coarse space method, such as these has been used by Cai, Gropp, Keyes and Tidriri [4] successfully for solving a nonlinear equation of aerodynamics.

1 Introduction

Considerable interest has developed in Schwarz type overlapping domain decomposition methods for the numerical solution of partial differential equations, see for examples [3, 6, 7, 11, 12, 16, 21, 22] and the references therein. This class of methods offers a great deal of parallelism and is very promising for modern parallel computers. The success of the methods depends heavily on the existence of an uniformly, or nearly uniformly, bounded decomposition of the function space in which the problem is defined. In this paper, we further enrich the Schwarz theory by providing a new technique of constructing an uniformly bounded decomposition of the problem space, which is more flexible and convenient for large, geometrically complicated, practical problems. It has a convergence rate that is similar to that of the regular Dryja-Widlund type decomposition [12], and does not require the coarse space to be a subspace of the original finite element space, in which the partial differential equation is discretized. Nor does it require that the collection of the un-extended subdomains forms a regular finite element subdivision.

We shall only discuss a two-level additive Schwarz algorithm, with a coarse and a fine grid. It is well known that the fine grid determines the accuracy of the discrete problem and that the only role of the coarse grid is to accelerate the convergence of the iterative method. In this paper, we try to minimize the inter-connection between the two grids by using a not necessarily nested coarse grid. As a result, the same coarse grid can be used even if the fine grid is locally refined, or re-meshed, to deal with the local singularity of the underlining problem. There are a number of ways to handle the communication between the two grids, in this paper, we insist on the computationally simplest one, i.e., pointwise interpolation. We show that this, sometimes troublesome, interpolation operator behaves well in both two- and three-dimensional space in our applications under certain reasonable assumptions on the two grids. For technical reasons, we assume that the coarse mesh is quasi-uniform, however, no such an assumption is needed for the fine mesh. Experiments with the pointwise interpolation in the context of non-nested multigrid methods can be found in [8, 17, 18].

In [13], Dryja and Widlund developed a general theory for Schwarz type algorithms which has a convergence rate characterized by the quantity $(1+H/\delta)$, where H measures the diameters of the subdomains (as well as the coarse mesh size) and δ the overlap between neighboring subdomains. This quantity indicates that subdomains with uniform aspect ratio is desired. In this paper, we develop a result involving

$$\min\{1+H_c^2/\delta^2,\ 1+H/\delta+H_c/\delta\cdot H_c/H\}, \tag{1}$$

where H_c is size of the coarse grid, which generally has nothing to do with the subdomain diameters H. The first quantity in (1) is independent of H. This allows us to use subdomains with arbitrary shape. As a consequence, our theory applies to the type of unstructured mesh problems decomposed by some graph-based partitioning techniques discussed by Cai and Saad in [5]. The second quantity in (1) reduces to that of Dryja and Widlund when $H \sim H_c$, and comes into play in the case of small overlap.

When solving system of equations arising from the discretization of non-selfadjoint, or indefinite, or nonlinear elliptic problems by Schwarz type algorithm, a fine enough coarse mesh space is usually necessary in order to make the convergence rate optimal, see e.g., [6, 7]. In such a case, using Dryja-Widlund construction [12] would normally result in a large number of subdomains that have to be combined later in order for the number of subproblems to fit into the number of processors of a parallel computer. With our new construction, the size of coarse mesh is totally independent of the number of subdomains.

In this paper, we shall focus only on a simple self-adjoint model problem, namely the homogeneous Dirichlet boundary value problem: Find $u \in V \equiv H_0^1(\Omega)$ such that

$$a(u,v)=(f,v), \quad \forall v \in V, \tag{2}$$

where the bilinear form $a(u,v)$ is defined by $a(u,v) = \int_\Omega \nabla u \nabla v dx$, $f(x) \in L^2(\Omega)$ is given, and Ω is an open bounded polygon in R^d ($d = 2$ or 3), with boundary $\partial\Omega$. We shall use $a(\cdot,\cdot)$ and $\|\cdot\|_a$ to denote the inner product and norm of $H_0^1(\Omega)$.

To introduce the finite element discretization, we let $\Omega_h = \{k_i\}$ be a standard finite element triangulation of Ω that satisfies the minimal angle condition, i.e., in two dimensional case

$$\gamma_k \geq \gamma_0 > 0, \quad \forall k \in \Omega_h,$$

where γ_k is the minimal interior angle of $k \in \Omega_h$ and γ_0 is a constant. We do not assume that the triangulation is quasi-uniform (see, e.g. [15]), i.e., that all elements are of nearly the same size. We allow the use of highly refined unstructured meshes. We define the corresponding finite element space $V_h \subset H_0^1(\Omega)$ as the regular piecewise linear continuous triangular finite element space on Ω. Let us denote by $\bar{h}$ as the maximum diameter of this finite element mesh which will be used later to restrict the size of the coarse grid.

The paper is organized as follows. In §2, we present a bounded decomposition of V_h constructed with a non-nested coarse mesh space. The boundedness is also established. The small overlap case is addressed separately in §3. An additive Schwarz algorithm is studied in §4. We also prove that its convergence rate is optimal. In §5 we provide a brief discussion of certain computational issues. An Appendix is devoted entirely to the issue of boundedness of the interpolation operator. In this paper, except in the Appendix, c and C, with or without subscripts, denote generic, strictly positive constants independent of any mesh parameters which will be introduced later.

2 A non-nested coarse mesh space

In this section, we describe a way of decomposing the domain Ω into a set of non-overlapping subdomains such that $\bar{\Omega} = \bigcup_i \bar{\Omega}_i$, and then into a set of overlapping subdomains, with which the Schwarz type algorithms is defined. In contrast to the Dryja-Widlund construction [12], we do not require that $\{\Omega_i\}$ forms a regular finite element subdivision of Ω. By adding a not-necessarily-nested coarse mesh space, we prove that uniformly bounded decomposition of V_h can be obtained. The use of non-nested spaces in multigrid methods has been studied in [2] and in domain decomposition methods in [9]. Other Schwarz algorithms that use nonconforming elements can be found in [20].

We begin by introducing several notations. Let $\Omega_{H_c} = \{\tau_i\}$ be a quasi-uniform triangulation of Ω and τ_i one of the triangles whose diameter is of order H_c. Ω_{H_c} will be referred to as the coarse grid. Here H_c is the maximum diameter of this coarse triangulation. We assume, throughout this paper, that

$$\bar{h} \leq const. H_c, \tag{3}$$

and that

$$area(k_i) \leq const. area(\tau_j), \quad \text{if } k_i \bigcap \tau_j \neq \emptyset. \tag{4}$$

Here $area(\cdot)$ means area in R^2 and volume in R^3.

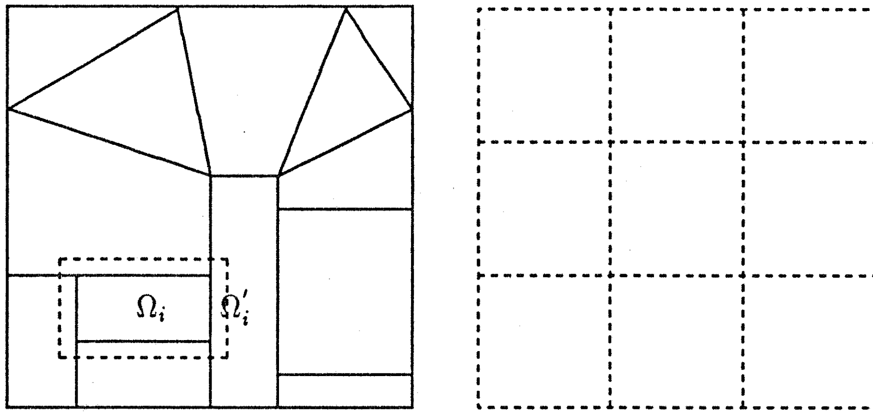

Figure 1: The left figure shows the partition of Ω into non-overlapping subdomains, which need not to form a regular finite element subdivision of Ω. The dotted line shows an extended subdomain. The right figure shows a coarse mesh, which is not necessarily nested with the fine mesh(not shown in the picture).

Let $V_{H_c} \subset H_0^1(\Omega)$ be a shape-regular finite element space over Ω consisting of piecewise linear continuous functions (cf. [10]). Note that, in general, $V_{H_c} \not\subset V_h$, and it is not necessary for V_{H_c} to have the same type of elements as V_h. For example, our theory holds if quadratic elements are used in V_{H_c} and linear triangular elements in V_h. Let $\Pi_h : C^0(\bar{\Omega}) \to V_h$ be the usual piecewise linear continuous interpolation operator, which uses values only at the nodal points of the fine mesh triangulation. We need the facts that Π_h satisfy the estimates provided in the following lemma.

Lemma 2.1 *There exists a positive constant C, independent of $\bar{h}$ and H_c, such that the following estimates hold in both two- and three-dimensional spaces.*
(i) $\|\Pi_h v\|_a \leq C\|v\|_a, \quad \forall v \in V_{H_c}$;
(ii) $\|v - \Pi_h v\|_{L^2(\Omega)} \leq C\bar{h}\|v\|_a, \quad \forall v \in V_{H_c}$.

We note that estimates (i) and (ii) normally do not hold if v is an arbitrary function in $L^2(\Omega)$ and $\Omega \in R^3$. However, we need the bounds only for functions in the subspace V_{H_c}. A proof of Lemma 2.1 will be given in the Appendix. Let

$$V_0 \equiv \Pi_h V_{H_c} \equiv \{v| \in V_h, \text{ there exists } w \in V_{H_c}, \text{ such that } v = \Pi_h w\},$$

which is a subspace of V_h. We shall use the L^2 projection operator $Q_{H_c} : H_0^1(\Omega) \to V_{H_c}$ defined by

$$(Q_{H_c} u, v) = (u, v), \quad \forall u \in H_0^1(\Omega), \ \forall v \in V_{H_c}.$$

We now partition Ω into non-overlapping subdomains $\{\Omega_i\}$, such that each $\partial\Omega_i$ does not cut through any elements in the finite element triangulation, and

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i \text{ and } \Omega_i \bigcap \Omega_j = \emptyset .$$

An example of such a partitioning is given in the left figure of Fig 1. Note that we do not assume that $\{\Omega_i\}$ forms a regular finite element subdivision of Ω, nor that the diameters of Ω_i are of the same order. In practice, a graph based partitioning technique, such as these introduced in [5, 14], can often be used to obtain Ω_i, especially if Ω_h is an unstructured grid. To obtain an overlapping decomposition of Ω, we extend each Ω_i to a larger subdomain $\Omega_i' \supset \Omega_i$, which is also assumed not cutting any fine mesh triangles, such that

$$\text{distance}(\partial\Omega_i' \cap \Omega, \partial\Omega_i \cap \Omega) \geq c\delta, \ \ \forall i,$$

for a constant $c > 0$. Here $\delta > 0$ will be referred to as the overlapping size. Associated with each Ω_i', we define a finite element space $V_i \equiv V_h \bigcap H_0^1(\Omega_i')$. In the next lemma, we prove that the decomposition

$$V_h = V_0 + V_1 + \cdots + V_N \tag{5}$$

exists, and is uniformly bounded.

Lemma 2.2 *For any $v \in V_h$, there exist $v_i \in V_i$, $i = 1, \cdots, N$ and $v_0' \in V_{H_c}$ such that*

$$v = \Pi_h v_0' + v_1 + \cdots + v_N, \tag{6}$$

and in addition, there exists a constant $C_0 > 0$ independent of the mesh parameters, such that

$$\|v_0'\|_a^2 + \sum_{i=1}^{N} \|v_i\|_a^2 \leq C_0 \left(1 + \frac{H_c^2}{\delta^2}\right) \|v\|_a^2, \quad \forall v \in V_h. \tag{7}$$

Proof. For any $v \in V_h$, we denote $v_0' = Q_{H_c} v \in V_{H_c}$, $v_0 = \Pi_h v_0' \in V_0$ and $w = v - v_0 \in V_h$. Because of the boundedness of Q_{H_c} in H_0^1 norm, we clearly have $\|v_0'\|_a \leq C\|v\|_a$. Let $\{\theta_i(x)\}$ be a partition of unity of Ω corresponding to $\{\Omega_i'\}$, such that $|\nabla\theta_i|_2 \leq C/\delta$ ($|\cdot|_2$ is the usual Euclidean norm in R^2 or R^3) and $\sum_{i=1}^N \theta_i(x) \equiv 1$. Of course, θ_i are smooth and $0 \leq \theta_i \leq 1$. We define $v_i = \Pi_h(\theta_i w) \in V_i$. It is easy to see that

$$v_0 + \sum_{i=1}^{N} v_i = v,$$

therefore (6) is proved.

Let k be a single triangular element in Ω_i' with diameter h. We assume that the average of θ_i over k is $\bar{\theta}_{i,k}$. It can be seen that

$$|v_i|_{H^1(k)}^2 \leq 2|\bar{\theta}_{i,k}\Pi_h w|_{H^1(k)}^2 + 2|\Pi_h\left((\theta_i - \bar{\theta}_{i,k})w\right)|_{H^1(k)}^2. \tag{8}$$

Because $\Pi_h w = w$, the first term on the right-hand side of the above inequality (8) presents no problems. We next estimate the second term. With the help of the element-wise inverse inequality, we have

$$\begin{aligned} |\Pi_h\left((\theta_i - \bar{\theta}_{i,k})w\right)|_{H^1(k)} &\leq C\frac{1}{h} \|\Pi_h\left((\theta_i - \bar{\theta}_{i,k})w\right)\|_{L^2(k)} \\ &\leq C\frac{1}{h}\frac{h}{\delta} \|w\|_{L^2(k)} = C\frac{1}{\delta} \|w\|_{L^2(k)}, \end{aligned} \tag{9}$$

where the constant $C > 0$ depends only on the finite element subdivision of Ω. The fact $|\theta_i - \bar{\theta}_{i,k}| \leq C\, h/\delta$ is also used. By taking the sum over all elements $k \in \Omega_i'$ we arrive at the estimate

$$|v_i|^2_{H^1(\Omega_i')} \leq C\left(|w|^2_{H^1(\Omega_i')} + \frac{1}{\delta^2}\|w\|^2_{L^2(\Omega_i')}\right), \tag{10}$$

which implies that

$$\sum_{i=1}^{N} |v_i|^2_{H^1(\Omega_i')} \leq C\left(\|w\|^2_a + \frac{1}{\delta^2}\|w\|^2_{L^2(\Omega)}\right). \tag{11}$$

Here the fact that each point in Ω is covered by only a finite number of overlapping subdomains is assumed. To bound the first term on the right-hand side of (11) in terms of $\|v\|_a$, we use the boundedness of the operators Π_h and Q_{H_c} in H_0^1 norm, i.e.,

$$\begin{aligned} \|w\|_a &= \|v - \Pi_h Q_{H_c} v\|_a \\ &\leq C\left(\|v\|_a + \|Q_{H_c} v\|_a\right) \leq C\|v\|_a. \end{aligned} \tag{12}$$

To estimate the second term on the right-hand side of (11) in terms of $\|v\|_a$, we need the L^2 regularity estimates of Π_h((ii) of Lemma 2.1) and Q_{H_c}, which give us

$$\begin{aligned} \|w\|_{L^2(\Omega)} &= \|v - \Pi_h Q_{H_c} v\|_{L^2(\Omega)} \\ &\leq \|v - Q_{H_c} v\|_{L^2(\Omega)} + \|Q_{H_c} v - \Pi_h Q_{H_c} v\|_{L^2(\Omega)} \\ &\leq C\left(H_c\,\|v\|_a + \bar{h}\,\|Q_{H_c} v\|_a\right) \leq C H_c\,\|v\|_a. \end{aligned} \tag{13}$$

Recall that $\bar{h} < H_c$ by assumption. The proof of the lemma thus follows immediately by combining the estimates (11), (12) and (13). □

We here make a few remarks about the bound (7) in Lemma 2.2. The bound in (7) depends only on the size of coarse grid and the overlapping size, and has nothing to do with the sizes, or diameters, of the subdomains. These un-extended subdomains $\{\Omega_i\}$ need not to form a finite element subdivision of Ω, and can be chosen to have any shape that best fit into a particular application. The estimate (7) is useful when the overlap between subdomains is generous, or roughly the size of the coarse mesh. If small overlap is preferred, a different estimate is given in the next section.

3 An estimate for the small overlap case

The decomposition bound (7) provided in the previous section grows at a rate proportional to $1/\delta^2$, which is rather large when small overlap is used. In this section, we discuss an alternative estimate of the bound, for the same decomposition (5) described in Lemma 2.2, and prove that it is in fact proportional only to $1/\delta$. Most of the techniques are borrowed from the recent paper of Dryja and Widlund [13]. As in the previous section, we do not assume that $\{\Omega_i\}$ forms a shape regular finite element triangulation of

Ω, however, for the small overlap case, we do need to assume that these $\Omega_i^{'}$s have roughly the same size, i.e., if H_i is the diameter of $\Omega_i^{'}$, then there exists a constant β, such that

$$1 \geq \frac{\min\{H_i\}}{\max\{H_i\}} \geq \beta.$$

Lemma 3.1 (Dryja and Widlund[13]) *Let $\Gamma_{\delta,i} \subset \Omega_i^{'}$ be the set of points that is within a distance of δ of $\partial\Omega_i \bigcap \Omega$, then*

$$\|w\|^2_{L^2(\Gamma_{\delta,i})} \leq C\delta^2 \left((1+\frac{H}{\delta})|w|^2_{H^1(\Omega_i^{'})} + \frac{1}{\delta H}\|w\|^2_{L^2(\Omega_i^{'})}\right), \quad \forall w \in H^1(\Omega_i^{'}), \ \forall i, \tag{14}$$

where H is the maximum diameter of these $\Omega_i^{'}$s. The constant C may depend on β.

Lemma 3.2 *The same decomposition (6), described in Lemma 2.2, exists and is bounded in the sense that there exists a constant $C_0 > 0$ independent of the mesh parameters, such that*

$$\|v_0^{'}\|_a^2 + \sum_{i=1}^{N}\|v_i\|_a^2 \leq C_0\left(1+\frac{H}{\delta}+\frac{H_c^2}{\delta H}\right)\|v\|_a^2, \quad \forall v \in V_h. \tag{15}$$

Proof. The proof is nearly the same as that for Lemma 2.2, except that we make use of the fact that $\theta_i - \bar{\theta}_{i,k} = 0$ if $k \subset \Omega_i^{'} \setminus \Gamma_{\delta,i}$. This implies that

$$|v_i|^2_{H^1(\Omega_i^{'})} \leq C\left(|w|^2_{H^1(\Omega_i^{'})} + \frac{1}{\delta^2}\|w\|^2_{L^2(\Gamma_{\delta,i})}\right). \tag{16}$$

By replacing the last term of (16) with the bound stated in Lemma 3.1, and summing over all subdomains, we obtain

$$\sum_{i=1}^{N}|v_i|^2_{H^1(\Omega_i^{'})} \leq C\left((1+\frac{H}{\delta})\|w\|_a^2 + \frac{1}{\delta H}\|w\|^2_{L^2(\Omega)}\right). \tag{17}$$

The proof is thus accomplished by using the inequalities (12) and (13). □

We note that the aspect ratio, or diameter, of the subdomains, H, appears in the the estimate in the case when small overlap is being used. The factor H is introduced into the estimate by Lemma 3.1. We do not know whether it can be removed, or replaced by certain quantity that is independent of the subdomain aspect ratio. We also comment that if the coarse mesh size H_c is the same as the diameter of the subdomains, then the result of Lemma 3.2 coincides with that of [13].

4 An additive Schwarz method based on a non-nested coarse space

In this section, we define and analyze an additive Schwarz algorithm for solving the finite element problem: Find $u_h^* \in V_h$ such that

$$a(u_h^*, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{18}$$

by using the subspace decomposition introduced in the previous sections. Let $V_h = V_0 + V_1 + \cdots + V_N$ be the decomposition discussed previously. For $1 \leq i \leq N$, we define the operator $P_i : V_h \rightarrow V_i$ by

$$a(P_i u, \phi) = a(u, \phi), \quad \forall u \in V_h \text{ and } \forall \phi \in V_i. \tag{19}$$

The definition of the space $V_0 = \Pi_h V_{H_c}$ implies that if $\{\psi_i\}$ are the basis functions of V_{H_c}, i.e., $V_{H_c} = span\{\psi_i\}$, then $V_0 = span\{\Pi_h \psi_i\}$. We define $P_0' : V_h \rightarrow V_{H_c}$ by

$$a(P_0' u, \phi) = a(u, \Pi_h \phi), \quad \forall u \in V_h \text{ and } \forall \phi \in V_{H_c} \tag{20}$$

and denote

$$P_0 = \Pi_h P_0' : V_h \rightarrow V_0. \tag{21}$$

Similar operators were used in the context of non-nested subspaces based multigrid methods, see for examples [1, 2]. In (20), if we choose $\phi = P_0' v \in V_{H_c}$, for an arbitrary $v \in V_h$, then the following identity holds

$$a(u, P_0 v) = a(P_0' u, P_0' v), \quad \forall u, v \in V_h,$$

which shows that the operator P_0 is symmetric with respect to the inner product $a(\cdot, \cdot)$. Let P be defined by

$$P = P_0 + P_1 + \cdots + P_N. \tag{22}$$

It can be seen easily that P is also symmetric in the inner product $a(\cdot, \cdot)$. Let $g = \Pi_h g_0 + \sum_{i=1}^N g_i$ and g_i be the solutions of the following subspace finite element problems: Find $g_0 \in V_{H_c}$ and $g_i \in V_i$ for $i \neq 0$, such that

$$a(g_i, \phi) = (f, \phi), \quad \forall \phi \in V_i \text{ and } i \geq 1$$

and

$$a(g_0, \phi) = (f, \Pi_h \phi), \quad \forall \phi \in V_{H_c}.$$

Following the Schwarz theory of Dryja and Widlund [12], it can be shown that, if the operator P is nonsingular, then the linear operator equation

$$P u_h^* = g \tag{23}$$

has the same solution as that of (18). We show in the next theorem that P is not only nonsingular but also uniformly bounded from both above and below.

Theorem 4.1 *The following estimate holds,*

$$c_0 \leq \|P\|_a \leq C_1, \tag{24}$$

where c_0 is the maximum of the following two constants

$$\frac{1}{C_0 (1 + H_c^2/\delta^2)},$$

which is defined in Lemma 2.2 and

$$\frac{1}{C_0\,(1 + H/\delta + H_c^2/\delta H)},$$

which is defined in Lemma 3.2. $C_1 > 0$ *is a constant independent of the parameters* h*,* H_c*,* δ *and* N*.*

Proof. The upper bound for P can be obtained trivially, we therefore omit its proof, see e.g., [12]. To estimate the lower bound, we look at only one of the cases corresponding to the constant $C_0(1 + H_c^2/\delta^2)$. The other case is nearly identical. Let $u \in V_h$, we first note that

$$\begin{aligned} a(u, Pu) &= a(u, \Pi_h P_0' u) + \sum_{i=1}^{N} a(u, P_i u) \\ &= a(P_0' u, P_0' u) + \sum_{i=1}^{N} a(P_i u, P_i u) \qquad (25) \\ &= \|P_0' u\|_a^2 + \sum_{i=1}^{N} \|P_i u\|_a^2. \end{aligned}$$

By using Lemma 2.2, we can write u as $u = \Pi_h u_0' + \sum_{i=1}^{N} u_i$, therefore,

$$\begin{aligned} a(u, u) &= a(u, \Pi_h u_0') + \sum_{i=1}^{N} a(u, u_i) \\ &= a(P_0' u, u_0') + \sum_{i=1}^{N} a(P_i u, u_i) \qquad (26) \\ &\leq \|P_0' u\|_a \|u_0'\|_a + \sum_{i=1}^{N} \|P_i u\|_a \|u_i\|_a \end{aligned}$$

With the Cauchy-Schwarz inequality, the boundedness estimate in Lemma 2.2 and (25) we have

$$\begin{aligned} a(u, u) &\leq \sqrt{\|P_0' u\|_a^2 + \sum_{i=1}^{N} \|P_i u\|_a^2}\,\sqrt{\|u_0'\|_a^2 + \sum_{i=1}^{N} \|u_i\|_a^2} \qquad (27) \\ &\leq \sqrt{a(u, Pu)}\,\sqrt{C_0(1 + \tfrac{H_c}{\delta})\ \|u\|_a^2} \end{aligned}$$

The lower bound of P can be obtained immediately by cancelling the common terms, $\|u\|_a$, and squaring both sides of the above inequality. □

The bounds stated in Theorem 4.1 show that the condition number of the operator P is independent of the mesh parameters. Thus, if an iterative method, such as the Conjugate Gradient method for the symmetric case and GMRES [19] for the nonsymmetric and indefinite case, is used to solve equation (23) then the number of iterations is independent of the mesh parameters.

5 Computational issues and final remarks

We now make a few remarks related to the computation of the coarse grid problem. Equation (23) is usually solved by an iterative method, which requires the computation of a matrix-vector multiply $Pu = \sum_{i=0}^{N} P_i u$ for some u in V_h at each iteration. The terms $P_i u$, $i \neq 0$, can be obtained with techniques described in several papers, see for example [12, 22]. Here we only briefly discuss how $P_0 u$ can be computed, i.e., how to setup and solve the coarse problem at each iteration. A complete discussion of the implementation issues can be found in a forthcoming book of Smith, Bjørstad and Gropp [22].

Let n and m be the dimensions of V_h and V_{H_c}, respectively, A and A_0 be the stiffness matrices of the fine and coarse spaces, respectively. If we let $\{x_i, i = 1, \cdots, n\}$ be the nodal points of the fine mesh with certain proper ordering, and $\{\psi_j(x), j = 1, \cdots, m\}$ the basis functions of V_{H_c}, we can define an $n \times m$ interpolation matrix, $E = (e_{ij})$, where $e_{ij} = \psi_j(x_i)$. Let $\underline{u}$ and $\underline{P_0 u}$ be the vector representations of u and $P_0 u$ in terms of the fine mesh basis functions, respectively, then $P_0 u$ can be computed with the formula

$$\underline{P_0 u} = E A_0^{-1} E^T A \underline{u},$$

which is simply a discretized version of (20) and (21).

Several numerical examples obtained by using this algorithm can be found in the paper of Cai and Saad [5], and the paper of Cai, Gropp, Keyes and Tidriri [4], and therefore are not included in this paper. The multiplicative version of the algorithm was also discussed and tested there. The extension of the theory to nonsymmetric and/or indefinite elliptic problems can be obtained easily with the techniques developed in [6, 7].

6 Appendix: The boundedness of Π_h

If the interpolation operator Π_h is considered as a map from the space $H_0^1(\Omega)$ to V_h, then it generally does not satisfy the bounds stated in Lemma 2.1, because of the failure of some Sobolev imbedding theorems [10]. In this Appendix, we prove that Π_h, restricted to a subspace $V_{H_c} \subset H_0^1(\Omega)$, is indeed bounded if the two grids satisfy certain assumptions discussed below as we move along in the proof. The case where V_{H_c} is replaced by a subspace consisting of piecewise continuous quadratic functions has been studied by Dryja and Widlund [13]. In a trivial case, when $V_{H_c} \subset V_h$ then (i) of Lemma 2.1 holds as an equality with $C = 1$, and (ii) holds with $C = 0$. Our basic philosophy for the proof of Lemma 2.1 is based on the observation that if the coarse grid is not too fine, then most fine triangles are not cut into smaller pieces by the coarse grid. And only a small number of fine triangles are cut into a small number of smaller pieces. For these un-cut fine triangles, Lemma 2.1 holds at the element level.

We now provide a proof for Lemma 2.1. Recall that $\Omega_{H_c} = \{\tau_i\}$ and $\Omega_h = \{k_i\}$ are the coarse and fine triangulations, respectively, and both satisfy the minimal angle assumption. As mentioned earlier, both V_{H_c} and V_h contain piecewise linear continuous functions. In the reminder of this section, we shall use *const.*, instead of the usual c or C, as a generic constant which is positive and independent of any mesh parameters.

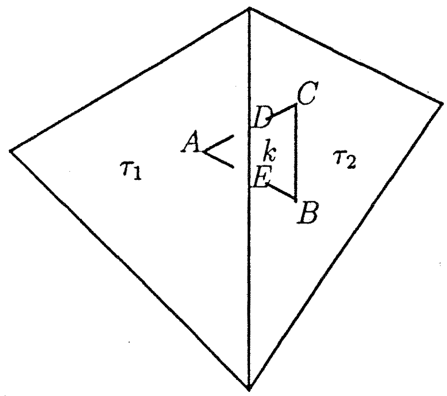

Figure 2: τ_1 and τ_2 belong to the coarse triangulation Ω_{H_c} and $k \in \Omega_h$ the fine triangulation. A, B and C are vertices of k. D and E are the intersection points as indicated.

We begin with part (i) of Lemma 2.1. The proof for the one dimensional case is trivial. We focus only on the two dimensional case in this paper. Slight modification is needed for the three dimensional, tetrahedra case. The essential step is to establish the estimate

$$|\Pi_h u|^2_{H^1(k)} \leq const. \left(\sum |\nabla u|_\tau|_2^2\right) area(k), \quad \forall u \in V_{H_c}, \tag{28}$$

where $k \in \Omega_h$, and the summation is taken over all coarse triangles $\tau \in \Omega_{H_c}$ that have non-empty intersection with k. As mentioned earlier, $area(k)$ denotes either the area or the volume of k. We note that $\nabla u|_\tau$ is a constant vector(since u is linear in τ) and $|\cdot|_2$ is the usual Euclidean norm in R^d.

If k belongs completely to a single τ, then (28) is obviously true. Without loss of generality, we assume that k intersects with only two coarse triangles τ_1 and τ_2, as shown in Fig 2. Let A, B, and C be the three vertices of k, and D, E be the intersection points as shown also in Fig 2. We shall use $\overline{AB}$, etc, to denote the distance between points A and B. We shall also use the elementary fact that

$$(\overline{AB})^2 + (\overline{AC})^2 + (\overline{BC})^2 \leq const.\ area(k), \tag{29}$$

where $const.$ depends only on the minimal angle of k.

Since we are considering linear elements, it is not difficult to show that

$$\begin{aligned} |\Pi_h u|^2_{H^1(k)} &\leq const.\, ((u(A)-u(B))^2 + (u(A)-u(C))^2 + (u(B)-u(C))^2) \\ &\leq const.\ [(u(A)-u(E))^2 + (u(A)-u(D))^2] + \\ &\quad const.\, [(u(D)-u(C))^2 + (u(E)-u(B))^2 + (u(B)-u(C))^2]. \end{aligned} \tag{30}$$

We note that points A, D and E are in τ_1, and B, C, D, E are in τ_2. By restricting ourselves to τ_1, we have

$$\begin{aligned}(u(A)-u(D))^2+(u(A)-u(E))^2 &\leq |\nabla u|_{\tau_1}|_2^2\,(\overline{AD})^2+|\nabla u|_{\tau_1}|_2^2\,(\overline{AE})^2 \\ &\leq |\nabla u|_{\tau_1}|_2^2\left((\overline{AB})^2+(\overline{AC})^2+(\overline{BC})^2\right) \qquad (31) \\ &\leq const.\,|\nabla u|_{\tau_1}|_2^2\, area(k),\end{aligned}$$

where the fact (29) is used in the last inequality. Similarly, we have, in τ_2, that

$$\begin{aligned}&(u(B)-u(C))^2+(u(B)-u(E))^2+(u(C)-u(D))^2 \\ &\leq const.\ |\nabla u|_{\tau_2}|_2^2\ area(k).\end{aligned} \qquad (32)$$

Therefore, estimate (28) is proved by combining the inequalities (30), (31) and (32).

For $\tau \in \Omega_{H_c}$, we denote by $\tau_j, j = 1, \cdots, l$ all the coarse triangles that share at least one of the fine triangles with τ (i.e., this fine triangle intersects with both τ and τ_j). We assume that l is a finite number. By summing (28) over all $k_i \in \Omega_h, i = 1, \cdots, m$, whose intersection with τ is non-empty, we obtain

$$\begin{aligned}|\Pi_h u|_{H^1(\tau)}^2 &\leq \sum_{i=1}^{m}|\Pi_h u|_{H^1(k_i)}^2 \\ &\leq const.\sum_{j=1}^{l}\left|\nabla u|_{\tau_j}\right|_2^2\ area(\tau_j).\end{aligned} \qquad (33)$$

Here we used that fact that, for each τ_j, the sum of the areas of the fine triangles that intersect with τ_j is less than $const.\,area(\tau_j)$, because of the assumption $\bar{h} \leq H_c$. The proof for part (i) of Lemma 2.1 follows immediately by summing (33) over all τ in Ω_{H_c} (the number of repetitions, for each τ, in the summation is finite).

We now turn to the proof of part (ii) of Lemma 2.1. It is sufficient to show that

$$\|w\|_{L^2(\Omega)}^2 \leq const.\ \bar{h}^2\ \left(|w|_{H_0^1(\Omega)}^2+|u|_{H_0^1(\Omega)}^2\right), \qquad (34)$$

for $w = u - \Pi_h u$ and $u \in V_{H_c}$. We note that w vanishes at all the fine nodal points, i.e., vertices of $k \in \Omega_h$.

Let $k_i \in \Omega_h$ and $\tau_j \in \Omega_{H_c}$, if $area(k_i \bigcap \tau_j) \neq 0$, we define the so-called subelement α_{ij} as $k_i \bigcap \tau_j$, see Fig 3 for example. We note that w is linear in any subelements. Furthermore, if α_{ij} contains at least one of k_i's vertices, we say α_{ij} is grounded (since w has a root in α_{ij}), otherwise, we say α_{ij} is un-grounded.

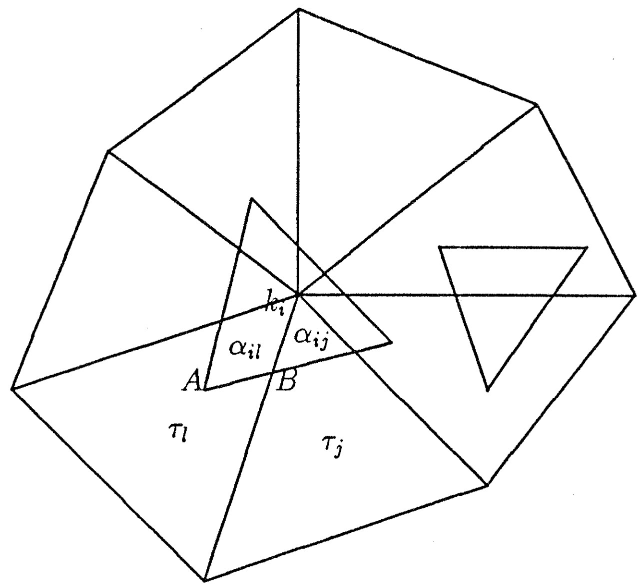

Figure 3: τ_j and τ_l are coarse elements, k_i is a fine element. The subelement $\alpha_{ij} = k_i \bigcap \tau_j$ is not grounded, but subelement $\alpha_{il} = k_i \bigcap \tau_l$ is grounded. Point A is the grounding point of α_{il} and B is a common point of α_{il} and α_{ij}.

We consider one subelement at a time. If α_{ij} is grounded, then there exists a point $A \in \alpha_{ij}$ such that $w(A) = 0$. Therefore,

$$\begin{aligned} \|w\|^2_{L^2(\alpha_{ij})} &= \int_{\alpha_{ij}} (w(x) - w(A))^2 \, dx \\ &\leq \int_{\alpha_{ij}} \left(|\nabla w|_2 \, |x - A|_2\right)^2 \, dx \\ &\leq \bar{h}^2 \, |w|^2_{H^1(\alpha_{ij})}. \end{aligned} \tag{35}$$

If α_{ij} is not grounded, such as what is shown in Fig 3, we take another subelement in the same k_i, say α_{il}, which is grounded, see also Fig 3. For simplicity, we assume that α_{ij} and α_{il} are adjacent to each other. Otherwise, one or maybe more intermediate subelements may be needed. We next prove that

$$\|w\|^2_{L^2(\alpha_{ij})} \leq const.\, \bar{h}^2 \left(|w|^2_{H^1(\alpha_{ij})} + |u|^2_{H^1(\tau_l)} + |\Pi_h u|^2_{H^1(k_i)} \right). \tag{36}$$

Let A be a grounding point of α_{il} and the point B belongs to both α_{ij} and α_{il}, then we have

$$\begin{aligned} \|w\|^2_{L^2(\alpha_{ij})} &= \int_{\alpha_{ij}} (w(x) - w(A))^2 \, dx \\ &\leq 2 \left(\int_{\alpha_{ij}} (w(x) - w(B))^2 \, dx + \int_{\alpha_{ij}} (w(B) - w(A))^2 \, dx \right) \\ &\leq 2 \left(\bar{h}^2 \, |w|^2_{H^1(\alpha_{ij})} + \bar{h}^2 \, |\nabla w|_{\alpha_{il}}|^2_2 \, area(\alpha_{ij}) \right). \end{aligned} \tag{37}$$

To bound the term $|\nabla w|_{\alpha_{il}}|_2^2 \; area(\alpha_{ij})$, we observe that $w = u - \Pi_h u$, and both u and $\Pi_h u$ are linear in α_{il}. Thus,

$$
\begin{aligned}
|\nabla w|_{\alpha_{il}}|_2^2 \; area(\alpha_{ij}) &\leq 2\,(|\nabla u|_{\alpha_{il}}|_2^2 + |\nabla \Pi_h u|_{\alpha_{il}}|_2^2) \; area(\alpha_{ij}) \\
&\leq const.\ (|\nabla u|_{\alpha_{il}}|_2^2 \, area(\tau_l) + |\nabla \Pi_h u|_{\alpha_{il}}|_2^2 \, area(k_i)) \\
&\leq const.\ \left(|u|_{H^1(\tau_l)}^2 + |\Pi_h u|_{H^1(k_i)}^2\right).
\end{aligned}
\tag{38}
$$

Here we made use of the facts that u is linear in τ_l and $\Pi_h u$ is linear in k_i, as well as the assumptions (3) and (4). The proof of (36) can thus be obtained by combining (37) and (38).

To complete the proof of (34), we take the sum of (35) and (36) over all possible subelements. Noting that for each coarse triangle, the number of neighboring un-grounded subelements is finite, therefore, we have

$$
\begin{aligned}
\|w\|_{L^2(\Omega)}^2 &\leq const.\ \bar{h}^2 \left(\sum_{\alpha_{ij}} |w|_{H^1(\alpha_{ij})}^2 + \sum_{\tau_l} |u|_{H^1(\tau_l)}^2 + \sum_{k_i} |\Pi_h u|_{H^1(k_i)}^2 \right) \\
&\leq const.\ \bar{h}^2 \left(|w|_{H_0^1(\Omega)}^2 + |u|_{H_0^1(\Omega)}^2 \right).
\end{aligned}
\tag{39}
$$

Here part (i) of Lemma 2.1 was used to bound the Π_h term in (39). The desired proof of part (ii) of Lemma 2.1 follows immediately.

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