# Epistemic Extension of Propositional Preference Logics Extended Version

Jacques Wainer

CU-CS-619-92 October 1992

University of Colorado at Boulder
DEPARTMENT OF COMPUTER SCIENCE

# Epistemic Extension of Propositional Preference Logics
## Extended Version

Jacques Wainer

Department of Computer Science
University of Colorado
Boulder, CO 80309
USA

phone: (303)-492-5911

e-mail: wainer@cs.colorado.edu

**Abstract**

Most of the current nonmonotonic logics are limited to a propositional or first-order language. This means that one cannot use these formalisms to model a nonmonotonic agent reasoning about the knowledge of other nonmonotonic agents, which limits the usefulness of such formalisms in modeling communication among agents.

This paper follows the approach that one should be able to extend some of the existing nonmonotonic logics to include formulas that contain a modal operator to denote the knowledge of other agents. We use a theory of utterance understanding as the source for our intuitions on what are the properties that such extended logics should exhibit.

The second part of this paper discusses a method to extend any propositional preference logics into a corresponding extended logics that allows for a knowledge operator. We then prove that the resulting logic satisfy all the requirement put forth in the first part.

**Keywords:** Knowledge Representation, logics for belief, nonmonotonic formalisms, multi-agent reasoning.

# 1 Introduction

Reasoning about other agents, and in particular reasoning about the beliefs of other agents, is of fundamental importance if an intelligent system is to deal with social situations. But the logics that have been created to deal with knowledge of more than one agent (for example [HM85]) have the limitation that the agents they model are monotonic. Since it is widely assumed that interesting forms of intelligence cannot be captured by monotonic forms of reasoning, these logics are very limited on their capacity of modeling interesting social behavior.

On the other hand, most of the existing nonmonotonic logics are limited to a first-order or a propositional language. That is, although these logics capture the nonmonotonicity of the agent's reasoning, they can only model the agent when it is reasoning about "things in the world," which can be expressed in either first-order or propositional languages. In particular, the existing nonmonotonic logics cannot model an agent reasoning about the knowledge of another agent.

Summarizing, the existing formal devices either model many "uninteresting" agents, or they can model only a single interesting agent. This paper addresses this problem: it describe a nonmonotonic logic that can model an agent reasoning about the knowledge of other nonmonotonic agents.

The approach taken in this paper is that one should be able to extend some of the existing non-monotonic logics to include formulas that refer to another agent's knowledge. We call these logics **epistemically extended**. This involves extending the semantics of such non-monotonic logics since most formalism (with the exception of default logic [Rei80]) are semantically limited to either propositional or first-order languages. But just allowing for formulas that contain a modal operator (to represent the knowledge of other agents) in the language does not by itself solves the problem of modeling nonmonotonic agents: the logic should yield the "correct" conclusions. This paper will also discuss what are the requirements that an epistemically extended logic should meet to derive the "correct" conclusions.

This paper is divided into two parts. The first part discusses the requirements that such an epistemic extended logic should meet in the context of a model of communication (or at least a model of utterance understanding). Section 2 describes a model of utterance understanding that solves some of the problems a naive theory would face. But this paper is not concerned with the consequences of this model, that is, this is not a paper on a model of utterance understanding. Instead we use the model to motivate the need of an epistemically extended logic, and to find out what are the requirements that the logic should meet.

The second part discusses a method of extending propositional preference logics into epistemic domains, and proves that the resulting logic meets the requirements put forth in the first part.

# 2 A Model of Communication

McCarthy [McC86] suggested that one of the many uses of nonmonotonic logics is to model conventions in communication. For example, the default rule "birds usually fly" can be seen as stating that if in a conversation a bird is mentioned and nothing is said about its flying condition, then one can assume that it flies. More specifically, if `S` (for speaker) tells `H` (for hearer) about a bird, and `S` says nothing about the bird's ability to flight, then `H` should conclude that the bird flies.

McCarthy [McC86] suggests that `H`'s reasoning can be done entirely within a first-order framework. This is done by representing the content of `S`'s assertion as a first-order formula in H's *belief space,* and combining it with H's beliefs about birds in general, and Tweety in particular. H would then perform the following nonmonotonic inference:

Tweety is a bird
Usually, birds fly

---

Tweety flies.

which McCarthy implements using circumscription.

We will call this method of modeling H's reasoning the **import-default** method, because the content of the utterance is first imported into H's belief space, and only after that are the defaults inferred. The import-default method has many shortcomings. First, it does not allow for the modeling of S's beliefs and the mutual beliefs between S and H . After S's statement that "Tweety is a bird," one can certainly conclude that S believes that Tweety flies, and that S (and H ) now assume that fact as part of their mutual knowledge. Thus, after that statement, either S or H can say "So the wall would not detain it" and expect the other to understand it.

A second shortcoming of the import-default method is the import process itself. If S's statement contradicts with H's beliefs, then H would not like to import it, and thus resulting in a contradictory knowledge base. For example, if H believes that Tweety is not a bird, but in fact something else, then after S's statement H should not import the content of the utterance into his knowledge base. Instead, H should conclude that S is mistaken about this attribute of Tweety (or that S is mistaken about the identity of Tweety). Furthermore, H should not import the content of S's statement if he has reason to believe that the statement contradict with S's beliefs. This includes the situations where S is lying (and H realizes it) or when S is being ironic.

Finally, the third shortcoming is that some utterances do not loose all their modal components when they are imported into the hearer's knowledge base. For a class of utterances that we named **epistemic cancellations**, the speaker uses the epistemic possibility operator to cancel (or block) the defaults. For example, by uttering

Tweety is a bird, perhaps a penguin.

the speaker blocks the default that Tweety flies. The semantic content of the utterance, even after it is imported into H's belief space, still carries the modal operator.

These shortcomings suggest that a more elaborate method to model understanding of utterances should be pursued. This method is based on explicitly reasoning about the speaker's beliefs, followed by a transferring step, where H accepts S's beliefs (or what he thinks are S's beliefs) as his own. This method is called **belief transfer** and was first discussed in [Per90]. The next section will discuss in more details the belief-transfer method, and show that in order to formalize it one needs nonmonotonic logics that can correctly deal with formulas containing a knowledge modal operator.

## 2.1 The belief transfer method

Like the import-default, the belief-transfer is a model of the hearer's reasoning process. But instead of importing the content of the utterance directly into the his own belief space and deriving the defaults in that space, the hearer derives the defaults in the speaker's belief space (or in his view of the speaker's belief space) and then transfer consistent beliefs from that space into his own. The belief transfer method is based on the following defaults:

- the speaker usually believes in what she says. This is Grice's maxim of quality [Gri75].

- if the speaker believes that a default holds, and that the antecedent of this default also holds, and that the consequence of the default does not contradicts with what else the speaker believes, then the speaker believes in the consequent of the default.
- if the hearer believes that the speaker believes in $p$ and $p$ does not contradicts with the hearer's beliefs, then the hearer should also believe $p$. This is the belief-transfer process.

We will now proceed to formalize these defaults in a suitable formal language.

### 2.1.1 Internal and External Logics for Belief

McArthur [McA88] discusses two different approaches to logics for belief: the internal and external points of view. The distinction is based on what is the meaning of asserting that a formula is true.

An internal logic assumes the point of view of the reasoner. A formula is true if the reasoner "knows" it. Thus, for an internal logic the formula:

$$bird(tweety)$$

means that the reasoner whose reasoning process the logic models, believes that Tweety is a bird.

An external logic assumes a "reality" point of view: asserting a formula as true means that it holds in reality. Thus, is an external logic, to refer to an agent's beliefs one has to explicitly use a belief operator. To assert that the agent H believes that Tweety is a bird, one has to assert the formula

$$\mathbf{B}_{\mathtt{H}}[bird(tweety)]$$

In this paper we will use an internal logic that models the hearer's reasoning process.[1] All formulas will implicitly refer to the hearer's knowledge and the inference rules of the logic will be abstractions or a model of the reasoning process that H performs. Thus, in the logic there will be no way of referring to the speaker's knowledge by itself, but only to the hearer's beliefs about the speaker knowledge. We will use the modal operator **B** to refer to (H's belief about) S's knowledge. Thus, the formula below:

$$p \wedge \mathbf{B}q$$

states that the hearer believes $p$ and he believes the speaker believes $q$.

Under the internal point of view, one can translate the basic defaults of the belief-transfer model in a formal language. We will use the symbol "$\leadsto$" as a generic representation of a default rule. Thus "$p \leadsto q$" represents the rule "$p$'s are usually $q$'s." The symbol "$\leadsto$" is a meta-level symbol that abbreviates the way a default rule is represented in a particular logic. For example, in circumscription, the default $p \leadsto q$ is implemented as $p \wedge \neg abn_1 \rightarrow q$ where $abn_1$ is one of the predicates being minimized. In autoepistemic logics, the same default would be represented as $p \wedge \neg\mathbf{L}\neg q \rightarrow q$ The symbol "$\vdash$" is the consequence relation of the nonmonotonic logic.

[1] The reason for this choice is that all nonmonotonic logics (with the exception of Levesque's logic of only-knowing [Lev90]) have always been internal: these logics attempt to model the reasoning process of a reasoner from the reasoner's own point of view. I would add that it is probably philosophically risky to define an external non-monotonic logic. Reasoning, and therefore internal logics which attempt to model it, is certainly nonmonotonic. But is "reality" nonmonotonic? A similar point was made by Levesque [Lev90] and Stalnaker [Sta92].

In an internal logic that models the hearer's reasoning process, the default rules in the belief-transfer model would be expressed as:

$$\mathbf{Say}\, p \ \vdash\ \mathbf{B}p \tag{1}$$

$$\mathbf{B}(p \rightsquigarrow q) \wedge \mathbf{B}p \ \vdash\ \mathbf{B}q \tag{2}$$

$$\mathbf{B}p \ \vdash\ p \tag{3}$$

Expression (1) is a representation of Grice's maxim of quality. It states that if H knows that S said something, he should assume that S believes in it. Expression (2) capture the hearer's belief that the speaker can perform nonmonotonic reasoning. And expression (3) is the belief transfer rule.

## 2.2 Goal of this paper

The goal of this paper is not to explore the belief-transfer model for utterance understanding, but to provide the tools for such an exploration. We are not interested in checking how well the axioms (1)–(3) capture the intuitions behind the belief transfer model, nor how well the belief transfer model itself models the process of utterance understanding. That will be the topic of another paper. Before that can be done, one needs to define what the default operator "$\rightsquigarrow$" and the nonmonotonic consequence relation "$\vdash$" mean in expressions (1)–(3). This is the goal of this paper.

Most nonmonotonic formalisms are limited to first-order or propositional languages. Therefore the entailments expressed in (2) and (3), are outside the scope of such formalisms. To be able to perform the reasoning described in (1)–(3), one must extend the existing formalisms in at least two directions. The first one is to allow defaults inside a knowledge operator, which is exemplified by expression (2). We call this extension **internal default**. Furthermore, (2) also illustrates that the internal default extension may involve two different aspects. The first one is related to the formula $\mathbf{B}(p \rightsquigarrow q)$ in (2): one should have enough syntactic devices to represent a default rule as a formula that can be inside the scope of the knowledge operator. This aspect is a problem for default logic [Rei80] because default rules are not formulas and therefore cannot be embedded inside a modal operator.

The second aspect of internal default is that the application of a default rule, that is, from $p \rightsquigarrow q$ and $p$, conclude $q$, should work as expected when these formulas are inside the scope of the knowledge operator, that is, from $\mathbf{B}[p \rightsquigarrow q]$ and $\mathbf{B}p$ conclude $\mathbf{B}q$.

The second direction in which the existing formalism must be extended is to allow for default rules whose arguments are modal formulas. We call this extension **external default**, and it is best illustrated when (3) is expressed as the application of a default rule:

$$(\mathbf{B}p \rightsquigarrow p) \wedge \mathbf{B}p \vdash p \tag{4}$$

The logic should allow for default rules to have one or more arguments that are modal formulas, and these default rules should derive the "correct" conclusions when they are applied.

To the author's knowledge, the belief transfer model is first discussed in [Per90]. Perrault formalizes the defaults in an external logic, using default logic [Rei80] to implement the nonmonotonicity. Default logic has no semantic limitations on implementing external default, and internal default is implemented in [Per90] as a meta-level rule that for each default rule $\frac{\alpha:\beta}{\beta}$ adds a rule $\frac{\mathbf{B}_i\alpha:\mathbf{B}_i\beta}{\mathbf{B}_i\beta}$. Epistemic cancellation is not discussed in that paper.

## 2.3 Assumptions

In this paper we will accept the following assumptions and simplifications:

1. We will not deal at all with expression (1). It introduces the operator **Say** whose properties are outside the scope of this paper. We are only concerned in extending some of the nonmonotonic formalisms to include the knowledge operator.

2. We will assume that the knowledge operator is a modal operator that follows the KD45 (or weak S5) axioms. The axioms are:

$$\begin{array}{rcl} \psi & & \text{if } \psi \text{ is a propositional tautology} \\ \mathbf{B}(\psi \rightarrow \phi) & \rightarrow & \mathbf{B}\psi \rightarrow \mathbf{B}\phi \\ \mathbf{B}\psi & \rightarrow & \neg\mathbf{B}\neg\psi \\ \mathbf{B}\psi & \rightarrow & \mathbf{B}\mathbf{B}\phi \\ \neg\mathbf{B}\psi & \rightarrow & \mathbf{B}\neg\mathbf{B}\psi \end{array}$$

   and the usual inference rules:

$$\textbf{MP:} \quad \frac{\psi \rightarrow \phi \qquad \psi}{\phi}$$

   and

$$\textbf{Nec:} \quad \frac{\vdash \psi}{\mathbf{B}\psi}$$

3. We will restrict the language to a propositional modal language.

4. We will not deal with multiple agents (besides the speaker and the hearer).

5. We will not deal with nested knowledge, that is, the speaker's belief about the hearer's beliefs, and so on. This is too strong a simplification if the goal of this paper were to study the belief transfer model. It is very likely that issues involving nested knowledge, and common knowledge are important in modeling communications.

We believe that although the assumptions made in this paper are too strong if the goal is to propose a model of communication, they can be incrementally weakened in order to construct more useful logics. For example, assumptions 4. and 5. were introduced to avoid multi-modalities, that is, more than a knowledge operator. In particular, that is the reason why nested knowledge is not discussed in this paper. To represent the speaker's belief about the hearer's belief one would need another modal operator. In this case, the formula $\mathbf{B}_{\mathtt{S}}\mathbf{B}_{\mathtt{H}}\alpha$ would represent that fact that (H believes that) S believes that H knows $\alpha$. We are currently investigating what are the requirements of such a multi-modalities logics.

## 2.4 Notation

In this paper we will use the following notations. The greek letters $\psi$, $\phi$, $\zeta$ denote formulas that may or not contain a modal operator (the operator **B** or it's dual **P**). The greek letters $\alpha$, $\beta$, $\gamma$, and $\delta$ denote propositional formulas, that is formulas without modal operators. The upper case greek letter $\Gamma$, and the letter $\mu$ denote sets (usually of models). The letters $p$, $q$ and so on denote propositional symbols.

## 2.5 Requirements for epistemically extended logics

In this section we define what are the requirements that the epistemically extended logics should meet. Among other things, we formalize the intuitions of the internal and external defaults discussed in section 2.2.

If $\mathcal{L}_X$ is a propositional nonmonotonic logic and $\vdash_X$ is the entailment or consequence relation of that logic, then we would like to define an epistemically extended logic $\mathcal{L}^*_X$, which extends the language of $\mathcal{L}_X$ to a modal propositional language, and also extends the entailment relation appropriately. $\vdash^*_X$ is the consequence or entailment relation of the logic $\mathcal{L}^*_X$.

The first requirement is that the logic $\mathcal{L}^*_X$ should have the same power as the logic $\mathcal{L}_X$ when dealing only with propositional formulas. This means that the logic $\mathcal{L}^*_X$ should indeed extend the logic $\mathcal{L}_X$ only when dealing with modal formulas. We call this requirement **extension** and we abbreviate it as **E**. Extension is captured formally as:

$$\alpha \vdash_X \beta \quad \text{if and only if} \quad \alpha \vdash^*_X \beta \tag{5}$$

The second requirement is that the logic $\mathcal{L}^*_X$ should include the logic chosen to represent knowledge. Since we use the logic KD45 to model knowledge, the logic $\mathcal{L}^*_X$ should be at least as powerful as KD45. We call this requirement **KD45-inclusion**, or **KD45i**, and it can be formalized as:

$$\text{if} \quad \psi \vdash_{KD45} \phi \quad \text{then} \quad \psi \vdash^*_X \phi \tag{6}$$

The logic $\mathcal{L}^*_X$ should also capture the mode of reasoning that we named internal default in (2). That is, we would expect that if a default rule can be applied in $\mathcal{L}_X$ it should be also applied in $\mathcal{L}^*_X$ when the defaults are inside the scope of the modal operator. We call this property of $\mathcal{L}^*_X$ as **internal default**, or **ID**. The formulation below extends the idea of defaults working inside the knowledge operator.

$$\alpha \vdash_X \beta \quad \text{if and only if} \quad \mathbf{B}\alpha \vdash^*_X \mathbf{B}\beta \tag{7}$$

If $\alpha$ above, contains both a default rule and its antecedent (for example $p \leadsto q$ and $p$) then it will correspond to the internal default as expressed in (2). But the formulation above also captures the interesting intuition that the hearer (whose reasoning the logic attempts to model) believes that speaker has the same (propositional) reasoning power as himself. If the hearer can deduce $\beta$ from $\alpha$, then he believes that if the the speaker believes $\alpha$ then she would also believe $\beta$.[2]

The forth requirement is related to external defaults. The logic $\mathcal{L}^*_X$ to be able to have default rules with formulas with modal operator as arguments, and these default rules should generate "correct" conclusions when they are applied. For example if

$$p \wedge (p \leadsto q) \vdash_X q$$

is an entailment of the logic $\mathcal{L}_X$, then both

$$\mathbf{B}p \wedge (\mathbf{B}p \leadsto q) \vdash^*_X q \qquad \text{and} \qquad p \wedge (p \leadsto \mathbf{B}q) \vdash^*_X \mathbf{B}q \tag{8}$$

should also be correct entailments in the logic $\mathcal{L}^*_X$.

It is somewhat difficult to capture the intuition behind external default formally. We will propose a formalization of external default that does not fully capture these intuitions, but is a step in that direction. A complete formalization still elude us.

[2] [MWC91] discusses an approach to modeling the beliefs of other agents based on this idea of attributing to others the same reasoning power as oneself and of explicitly reasoning about one's reasoning process.

The weak characterization of the external default requirement, abbreviated as **WED**, is an extension of (8), when $p$ and $q$ are general propositional formulas. That is:

$$\text{if} \quad \alpha \wedge (\alpha \rightsquigarrow \beta) \vdash_X \beta \quad \text{then} \quad \mathbf{B}\alpha \wedge (\mathbf{B}\alpha \rightsquigarrow \beta) \vdash_X^* \beta \quad \text{and} \quad \alpha \wedge (\alpha \rightsquigarrow \mathbf{B}\beta) \vdash_X^* \mathbf{B}\beta \tag{9}$$

Expression (9) above does not capture the full intuition behind external default because, for example, it does not deal with conflicting defaults. If

$$p \wedge (p \rightarrow b) \wedge (p \rightsquigarrow \neg f) \wedge (b \rightsquigarrow f) \vdash_X \neg f$$

then we would like that

$$p \wedge (p \rightarrow \mathbf{B}b) \wedge (p \rightsquigarrow \neg f) \wedge (\mathbf{B}b \rightsquigarrow f) \vdash_X \neg f$$

This is not captured by (9).

Finally, the next requirements are related to epistemic cancellation and the intuitions behind them require some further elaboration.

### 2.5.1 Epistemic Cancellation

As mentioned above, epistemic cancellation are a class of utterances in which the speaker uses the epistemic possibility operator to cancel or block a default that would otherwise be attributed to her. For example, if the speaker had uttered:

Tweety is a bird.

and given the default that birds usually fly, the hearer should conclude that the speaker knows that Tweety flies. Epistemic cancellation is a way of canceling this knowledge attribution by explicitly saying that the speaker believes it to be possible that the default would not hold in this case. Thus by uttering

Tweety is a bird. Perhaps a penguin. (10)

the speaker is explicitly saying that she considers it possible that Tweety is a penguin and therefore that Tweety cannot fly. This blocks the conclusion that the speaker believes that Tweety could fly.

This intuition is captured by the following requirement for the extended logic $\mathcal{L}_X^*$.

$$\text{if} \quad \alpha \vdash_X \beta \quad \text{and} \quad \alpha \wedge \delta \not\vdash_X \beta \quad \text{then} \quad \mathbf{B}\alpha \wedge \mathbf{P}\delta \not\vdash_X^* \mathbf{B}\beta \tag{11}$$

The only aspect of (11) that needs to be commented is the formula $\mathbf{B}\alpha \wedge \mathbf{P}\delta$. We are assuming that the speaker believes in what she says. Thus, by uttering (10) one would state that the speaker knows the content of the utterance, that is:

$$\mathbf{B}[\, bird(Tweety) \wedge \mathbf{P}penguin(Tweety)\,]$$

Distributing the knowledge operator, and given that $\mathbf{BP}\alpha \leftrightarrow \mathbf{P}\alpha$ is a theorem of the logic **KD45** or **S5**, results in a formula in the same form of (11):

$$\mathbf{B}bird(Tweety) \wedge \mathbf{P}penguin(Tweety)$$

The requirement above is called **weak epistemic cancellation** (**WEC**). Weak because it specifies only what should not be derivable in the extended logic, but does not specify what should be derivable in the case of epistemic cancellation. The next two requirements specify what should be derivable in epistemic cancellation situations. The first one is the **irrelevant epistemic cancellation** (or **IEC**) and it is based on the idea that if the epistemic possibility clause do not cancel a default, then the default should be derivable. In other words, if the speaker had said:

Tweety is a bird. Perhaps a sparrow.

and since sparrows usually fly, then one should conclude that Tweety flies. Formally:

$$\text{if} \quad \alpha \vdash_X \beta \quad \text{and} \quad \alpha \wedge \delta \vdash_X \beta \quad \text{then} \quad \mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash_X^* \mathbf{B}\beta \tag{12}$$

The final requirement addresses the issue of what should be concluded about Tweety's flying ability when (10) is uttered. One position is that nothing can be concluded from (10). One could claim that because the speaker expressed her doubts about whether Tweety is a penguin or not, one cannot conclude anything about Tweety's flying abilities. A second position would claims that the corresponding defaults do apply to each of the possibilities raised by the speaker. In the example above, the possibilities are that Tweety is a penguin, and that Tweety is a non-penguin bird. For each of these possibilities the relevant defaults should apply. If Tweety is a penguin, then it does not fly (because by default penguins do not fly), and if Tweety is a non-penguin bird then it should fly. Thus, this second view would claim that the conclusion one should derive from the utterance of (10) is that either Tweety is a non-flying penguin, or Tweety is a flying, non-penguin bird. We believe that this second position, that defaults holds for each epistemic possibility raised by the speaker, is the correct one.

This second view is the motivation for the last requirement on the logic $\mathcal{L}_X^*$, the **strong epistemic cancellation(SEC)**. It states:

$$\text{if} \quad \alpha \vdash_X \beta \quad \text{and} \quad \alpha \wedge \delta \vdash_X \gamma \quad \text{then} \quad \mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash_X^* \mathbf{B}[(\beta \wedge \neg\delta) \vee (\delta \wedge \gamma)] \tag{13}$$

For example, to deal with example (10), one take $\alpha$ to stand for the conjunction of all the relevant knowledge about birds and penguins, and the statement $bird(tweety)$; $\beta$ stands for $fly(tweety)$; $\delta$ stands for $penguin(tweety)$; and $\gamma$ stands for $\neg fly(tweety)$.

These requirements are not independent. We show below that SEC in some way includes both IEC and WEC.

**Theorem 1** *SEC* $\rightarrow$ *IEC*

**Proof 1.** IEC is a special case of SEC when $\gamma$ is $\beta$. In this case we would have

$$\begin{array}{ll} \text{if} & \alpha \vdash_X \beta \\ & \alpha \wedge \delta \vdash_X \beta \\ \text{then} & \mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash_X^* \mathbf{B}[(\beta \wedge \neg\delta) \vee (\beta \wedge \delta)] \\ \text{or} & \mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash_X^* \mathbf{B}\beta \end{array}$$

which is the expression for IEC.

∎

**Theorem 2** *SEC + KD45i* $\rightarrow$ *WEC*

**Proof 2.** WEC derives from SEC when $\gamma \rightarrow \neg\beta$ is a tautology, (that is if $\gamma$ and $\beta$ are contradictory). Then, because we assume that the logic $\mathcal{L}$ is not contradictory:

$$\alpha \wedge \delta \not\vdash_X \beta$$

which is the precondition to WEC.

The proof proceeds by contradiction. Let us suppose that both

$$\mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash_X^* \mathbf{B}[(\beta \wedge \neg\delta) \vee (\gamma \wedge \delta)] \tag{14}$$

$$\mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash_X^* \mathbf{B}\beta \tag{15}$$

Since $\gamma \rightarrow \neg\beta$ is a tautology, then by KD45-i:

$$\vdash^*_X \mathbf{B}(\gamma \rightarrow \neg\beta) \tag{16}$$

Together (14), (15) and (16) imply that

$$\mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash^*_X \mathbf{B}[\beta \wedge \neg\delta]$$

which in turn implies

$$\mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash^*_X \mathbf{B}\neg\delta \tag{17}$$

But if $\vdash^*_X$ has the property of KD45 inclusion, the expression in (17) is contradictory since $\mathbf{P}\delta$ and $\mathbf{B}\neg\delta$ are contradictory.

∎

### 2.6 Summary

For convenience, let us summarize the requirements on the logic $\mathcal{L}^*_X$. If $\vdash_X$ is the entailment or consequence relation of the propositional nonmonotonic logic $\mathcal{L}_X$, then the epistemic extension of that logic, defined by the new entailment $\vdash^*_X$ should meet the following requirements:

| | |
|---|---|
| **E:** | $\alpha \vdash_X \beta$ if and only if $\alpha \vdash^*_X \beta$ |
| **KD45i:** | if $\psi \vdash_{KD45} \phi$ then $\psi \vdash^*_X \phi$ |
| **ID:** | $\alpha \vdash_X \beta$ if and only if $\mathbf{B}\alpha \vdash^*_X \mathbf{B}\beta$ |
| **ED:** | if $\alpha \vdash_X \beta$ then $(S_i\alpha) \vdash^*_X (S_i\beta)$ for all substitutions $S_i$ |
| **WEC:** | if $\alpha \vdash_X \beta$ and $\alpha \wedge \delta \not\vdash_X \beta$ then $\mathbf{B}\alpha \wedge \mathbf{P}\delta \not\vdash^*_X \mathbf{B}\beta$ |
| **IEC:** | if $\alpha \vdash_X \beta$ and $\alpha \wedge \delta \vdash_X \beta$ then $\mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash^*_X \mathbf{B}\beta$ |
| **SEC:** | [t]if $\alpha \vdash_X \beta$ and $\alpha \wedge \delta \vdash_X \gamma$ then $\mathbf{B}\alpha \wedge \mathbf{P}\delta \vdash^*_X \mathbf{B}[(\neg\delta \wedge \beta) \vee (\delta \wedge \gamma)]$ |

## 3 Epistemic Extension of Preference Logics

In this section we will describe the epistemic extensions of propositional preference logics. Or more precisely, we will describe a method of defining the epistemic extension of any particular preference logic. The work described here is based on [Wai92b]. The main differences are with [Wai92b] is that the definition of elementary improvement is changed (so that the resulting logic satisfies external default), and we prove that the resulting logic has all the properties put forth in the previous section.

In model preference logics, which include most of the forms of circumscription, the definition of entailment is based on a partial order among the models of the theory [Lif85, Sho87]. Given a partial order "$\leq$" among models, one defines entailment as the propositions that are satisfiable by the $\leq$-minimal models of the theory. Formally:

$$\begin{array}{lcl} \Gamma_0(\psi) & = & \{M \mid M \models \psi\} \\ \Gamma_1(\psi, \leq) & = & \{M \mid M \in \Gamma_0(\psi) \text{ and } \neg\exists \overline{M} \in \Gamma_0(\psi) \text{ such that } \\ & & \qquad \overline{M} \leq M \text{ and } \overline{M} \neq M\} \\ \psi \models_\leq \phi & \text{iff} & \forall M,\ M \in \Gamma_1(\psi, \leq) \text{ implies } M \models \phi \end{array} \tag{18}$$

A **propositional-model**, that is, a model for a formula restricted to a propositional language, is a valuation function $w$ that assigns a truth value to all propositional symbols in the language. The truth value that the propositional-model assigns to a compound formula is defined by the usual recursive rules. If a propositional model assigns **true** to a formula $\alpha$, we will say that the model satisfy the formula, and denote it by the notation $w \models \alpha$.

The epistemic extension of preference logic will also be a preference logic characterized by the partial-order relation $\sqsubseteq$ among modal-models (that is models for formulas from a modal language). And the partial order $\sqsubseteq$ will be based on the original partial order $\leq$. The next section describes the construction of the partial order $\sqsubseteq$, and the following one proves that the logic defined by the modal partial relation $\sqsubseteq$ satisfy all requirements in the previous section.

## 3.1 The definition of $\models_{\sqsubseteq}$

A **KD45-model** is a tuple $\langle w_0, W\rangle$ where $w_0$ a propositional-model, named the *real world*, and $W$ is a set of propositional-models. Each of the elements of $W$ and $w_0$ are called ***worlds***. The satisfiability relation for KD45-models is defined as usual:

$$\begin{array}{llll}
\langle w_0, W\rangle & \models & \alpha & \text{iff} \quad w \models \alpha \\
\langle w_0, W\rangle & \models & \psi \wedge \phi & \text{iff} \quad \langle w_0, W\rangle \models \psi \quad \text{and} \quad \langle w_0, W\rangle \models \phi \\
\langle w_0, W\rangle & \models & \neg\psi & \text{iff} \quad \langle w_0, W\rangle \not\models \psi \\
\langle w_0, W\rangle & \models & \mathbf{B}\psi & \text{iff} \quad \text{for all } w \in W,\ \langle w, W\rangle \models \psi
\end{array}$$

We will now define a auxiliary relation $\sqsubseteq_e$ among KD45-models based on the $\leq$ relation among propositional models. Given two KD45-models $M_1 = \langle w_{0_1}, W_1\rangle$ and $M_2 = \langle w_{0_2}, W_2\rangle$, we will say that $M_1$ is an **elementary improvement** of $M_2$, or $M_1 \sqsubseteq_e M_2$, if:

$$\begin{array}{l}
w_{0_1} \leq w_{0_2} \quad \text{or} \\
W_1 = W_2 \quad \text{or} \\
W_2 = W_1 \cup \{\overline{w}\} \text{ and there exists } w' \in W_1 \text{ and } w' \leq \overline{w} \quad \text{or} \\
W_2 = Q \cup \{w_2\} \text{ and } W_1 = Q \cup \{w_1\} \text{ and } w_1 \leq w_2, \text{ for some set } Q
\end{array} \tag{19}$$

Intuitively, $M_1$ is an elementary improvement of $M_2$ if the real world in $M_1$ is "smaller" (in the $\leq$ sense) than the real world in $M_2$, or if $W_2$ has one world more than $W_1$ and there is a world in $W_1$ that is smaller than the missing world, or if $W_1$ and $W_2$ disagree in only one world and that extra world in $W_1$ is smaller than the extra world in $W_2$.

The partial order $\sqsubseteq$ is defined as the transitive closure of $\sqsubseteq_e$. Finally, the entailment relation $\models_{\sqsubseteq}$ is defined as:

$$\begin{array}{lll}
\Gamma_0(\psi) & = & \{M \mid M \models \psi\} \\
\Gamma_1(\psi, \sqsubseteq) & = & \{M \mid M \in \Gamma_0(\psi) \text{ and } \neg\exists \overline{M} \in \Gamma_0(\psi) \text{ such that} \\
 & & \qquad \overline{M} \sqsubseteq M \text{ and } \overline{M} \neq M\} \\
\psi \models_{\sqsubseteq} \phi & \text{if} & \forall M,\ M \in \Gamma_1(\psi, \sqsubseteq) \text{ implies } M \models \phi
\end{array} \tag{20}$$

The definition above is similar to (18) with the exception that $\sqsubseteq$ is used instead of $\leq$, $M$ in the definition above are KD45-models, and $\models$ is the satisfiability relation for KD45-models.

The next section will prove that the logic defined by $\models_{\sqsubseteq}$ does satisfy the requirements put forth in section 2.5.

We will use the following abbreviations, where $\alpha$ is a propositional formula:

$$\Gamma_0(\psi) = \{\langle w, W\rangle | \langle w, W\rangle \models \psi\} \tag{21}$$
$$\Gamma_1(\psi) = \{M | M \in \Gamma_0(\psi) \text{ and } \neg\exists M' \in \Gamma_0(\psi) \wedge M' \sqsubset M\} \tag{22}$$
$$\mu_0(\alpha) = \{w | w \models \alpha\} \tag{23}$$
$$\mu_1(\alpha) = \{w | w \in \mu_0(\alpha) \text{ and } \neg\exists w' \in \mu_0(\alpha) \wedge w' < w\} \tag{24}$$

That is, for a possibly modal formula $\psi$, $\Gamma_0(\psi)$ collects all KD45-models that satisfy $\psi$; $\Gamma_1(\psi)$ collects all $\sqsubseteq$-minimal KD45-models that satisfy $\psi$. For a propositional formula $\alpha$, $\mu_0(\alpha)$ collects all propositional models that satisfy $\alpha$, and $\mu_1(\alpha)$ collects all $\leq$-minimal propositional-models that satisfy $\alpha$.

## 3.2 Extension holds for $\models_{\sqsubseteq}$

**Theorem 3 (E)** *For $\alpha$ and $\beta$ propositional: $\alpha \models_{\leq} \beta$ if and only if $\alpha \models_{\sqsubseteq} \beta$.*

**Proof 3.** By contradiction for both directions.

- $\rightarrow$ **direction.** Assume that $\alpha \models_{\leq} \beta$ and that $\alpha \not\models_{\sqsubseteq} \beta$. Then there is a model $\overline{M} = \langle \overline{w}, \overline{W} \rangle$ such that $\overline{M} \in \Gamma_1(\alpha)$ and $\overline{M} \not\models \beta$. On the other hand, because $\alpha \models_{\sqsubseteq} \beta$, all $\leq$-minimal models of $\alpha$ satisfy $\beta$, therefore $\overline{w} \notin \mu_1(\alpha)$. Since $\overline{w} \in \mu_0(\alpha)$, then there exists a model $\overline{\overline{w}} \in \mu_1(\alpha)$ such that $\overline{\overline{w}} < \overline{w}$. But then the model $\overline{\overline{M}} = \langle \overline{\overline{w}}, \overline{W} \rangle$, will also satisfy $\alpha$, and $\overline{\overline{M}} \sqsubset \overline{M}$, which contradicts the assumption that $\overline{M} \in \Gamma_1(\alpha)$.
- $\rightarrow$ **direction.** Assume that $\alpha \not\models_{\leq} \beta$ and $\alpha \models_{\sqsubseteq} \beta$. Then there is a propositional-model $\overline{w} \in \mu_1(\alpha)$ such that $\overline{w} \not\models \beta$. On the other hand, all $\sqsubseteq$-minimal model of $\alpha$ satisfy $\beta$. But then the KD45-model $M = \langle \overline{w}, \emptyset \rangle$ belongs to $\Gamma_0(\alpha)$ and are $\sqsubseteq$-minimal, but does not satisfy $\beta$. This contradicts with the assumption that all models in $\Gamma_1(\alpha)$ satisfy $\beta$.

∎

## 3.3 KD45-inclusion holds for $\models_{\sqsubseteq}$

**Theorem 4 (KD45-i)** *If $\psi \models_{KD45} \phi$ then $\psi \models_{\sqsubseteq} \phi$.*

**Proof 4.** Since all KD45-models are models for the modal logic KD45, and since each KD45-models that satisfy $\psi$ also satisfy $\phi$, then each $\sqsubseteq$-minimal models that satisfy $\psi$ will also satisfy $\phi$.

∎

## 3.4 Internal default holds for $\models_{\sqsubseteq}$

In order to further simplify the notation, we will abbreviate $\Gamma_0(\mathbf{B}\alpha)$ by $\Gamma_0$, $\Gamma_1(\mathbf{B}\alpha)$ by $\Gamma_1$, $\mu_0(\alpha)$ by $\mu_0$, and $\mu_1(\alpha)$ by $\mu_1$.

The lemma below will relate the sets $\Gamma_1$ and $\mu_1$.

**Lemma 5** $\Gamma_1 = \{\langle w, W \rangle \mid W \subseteq \mu_1(\alpha)$ *and* $\neg\exists w',\ w' < w\}$

**Proof 5.**

- $\supseteq$ **direction.** Let us take $\overline{M} = \langle \overline{w}, \overline{W} \rangle \in \{\langle w, W \rangle \mid W \subseteq \mu_1$ and $\neg\exists w',\ w' < w\}$ and prove that $\overline{M} \in \Gamma_1$. $\langle \overline{w}, \overline{W} \rangle \models \mathbf{B}\alpha$ since $\overline{W} \subseteq \mu_1$ and thus, $\forall w \in \overline{W},\ w \models \alpha$. Therefore $\overline{M} \in \Gamma_0$. Now let us prove that $\overline{M}$ is also $\sqsubseteq$-minimal, that is, that there is no model $\overline{\overline{M}} = \langle \overline{\overline{w}}, \overline{\overline{W}} \rangle$ in $\Gamma_0$ such that $\overline{\overline{M}} \sqsubset \overline{M}$. By contradiction. $\overline{\overline{M}} \sqsubset \overline{M}$ if $\overline{\overline{w}} < \overline{w}$ which contradicts with $\langle \overline{w}, \overline{W} \rangle \in \{\langle w, W \rangle | W \subseteq \mu_1(\alpha)$ and $\neg\exists w',\ w' < w\}$. Or, $\overline{\overline{M}} \sqsubset \overline{M}$ if $\exists \overline{w_1} \in \overline{W}$ and $\exists \overline{\overline{w}}_2 \in \overline{\overline{W}}$ and $\overline{\overline{w}}_2 < \overline{w}_1$. But then, $\overline{\overline{w}}_2 \models \alpha$ and $\overline{w}_1 \models \alpha$, which contradicts with $\overline{w}_1 \in \overline{W} \subseteq \mu_1$.

- $\subseteq$ **direction.** By contradiction, let us take $\overline{M} = \langle \overline{w}, \overline{W} \rangle \in \mu_1$ and assume it does not belong to $\{\langle w, W\rangle | W \subseteq \mu_1(\alpha)$ and $\neg\exists w',\ w' < w\}$. Then either a) $\exists\overline{\overline{w}} < \overline{w}$, or b) $\exists \overline{w_1} \in \overline{W}$, and $\overline{w_1} \notin \mu_1$. If a) then lets take the model $\overline{\overline{M}} = \langle \overline{\overline{w}}, \overline{W}\rangle$. Since $\overline{\overline{w}} < \overline{w}$, then $\overline{\overline{M}} \sqsubset \overline{M}$, and since $\overline{\overline{W}} \models \mathbf{B}\alpha$, then $\overline{M}$ is not $\sqsubseteq$-minimal, which contradicts the assumption $\overline{M} \in \Gamma_1$. If b) then: $\overline{w}_1 \in \mu_0$ since $\overline{M} \models \mathbf{B}\alpha$. Then there is $\overline{\overline{w}} \in \mu_1$ such that $\overline{\overline{w}} < \overline{w}_1$. Now let us consider the model $\overline{\overline{M}} = \langle \overline{w}, \overline{W} - \{\overline{w}_1\} + \{\overline{\overline{w}}\}\rangle$. $\overline{\overline{M}} \models \mathbf{B}\alpha$, and thus $\overline{\overline{M}} \in \Gamma_0$. Also $\overline{\overline{M}} \sqsubset \overline{M}$, which contradicts with the assumption that $\overline{M} \in \Gamma_1$.

  ∎

**Theorem 6 (ID)** *$\alpha \models_{\leq} \beta$ if and only if $\mathbf{B}\alpha \models_{\sqsubseteq} \mathbf{B}\beta$.*

**Proof 6.** Given the definitions of the sets $\mu_1$ and $\Gamma_1$, and the definition of the entailment relations $\models_{\leq}$ and $\models_{\sqsubseteq}$, the claim of the theorem is equivalent to

$$\forall w \in \mu_1,\ w \models \beta \quad \text{iff} \quad \forall M \in \Gamma_1,\ M \models \mathbf{B}\beta$$

- $\rightarrow$ **direction.** By lemma 5, a model $M \in \Gamma_1$ is of the form $\langle w, W\rangle$ where $W \subseteq \mu_1$. Since all worlds in $\mu_1$ satisfy $\beta$, so do all worlds in $W$. Therefore $M \models \mathbf{B}\beta$.

- $\leftarrow$ **direction.** Since all models in $\Gamma_1$ satisfy $\mathbf{B}\beta$, let us pick a model $M = \langle w, \mu_1\rangle$. Since $M \models \mathbf{B}\beta$, then all words in $\mu_1$ satisfy $\beta$.

  ∎

## 3.5 External-default holds for $\models_{\sqsubseteq}$

To prove that weak external default holds for any particular logic one has to be specific about how that logic implements a default rule. We will prove that WED holds for propositional circumscription. In propositional circumscription, one represents the default $\alpha \rightsquigarrow \beta$ as $\alpha \wedge ab_1 \rightarrow \beta$ where $ab_1$ should be one of the propositional symbols to be minimized (that is the preference relation $\leq$ all other things being equal should prefer a model where $ab_1$ is false).

**Theorem 7 (WED)** *If $\alpha \wedge (\alpha \wedge \neg ab_1 \rightarrow \beta) \models_{\leq} \beta$ when $\leq$ (also) minimizes $ab_1$, then both $\mathbf{B}\alpha \wedge (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta) \models_{\sqsubseteq} \beta$ and $\alpha \wedge (\alpha \wedge \neg ab_1 \rightarrow \mathbf{B}\beta) \models_{\sqsubseteq} \mathbf{B}\beta$*

**Proof 7.**

- Let us first deal with $\mathbf{B}\alpha \wedge (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta)$. By contradiction, let us assume that $\mathbf{B}\alpha \wedge (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta) \not\models_{\sqsubseteq} \beta$. That is, there is a model $M = \langle w_0, W\rangle$ in $\Gamma_1(\mathbf{B}\alpha \wedge (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta))$ such that $w_0 \not\models \beta$. But since $M \models \mathbf{B}\alpha$ and $M \models (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta)$, it follows that $w_0 \models ab_1$. Since $\leq$ also minimizes $ab_1$, there is a world $\overline{w}_0$ such that $\overline{w}_0 \leq w_0$, and thus there is model $\overline{M} = \langle \overline{w}_0, W\rangle$, such that $\overline{M} \models \mathbf{B}\alpha \wedge (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta)$ and $\overline{M} \sqsubset M$, which contradicts with the fact that $M \in \Gamma_1(\mathbf{B}\alpha \wedge (\mathbf{B}\alpha \wedge \neg ab_1 \rightarrow \beta))$

- Now lets us deal with $\alpha \wedge (\alpha \wedge \neg ab_1 \rightarrow \mathbf{B}\beta)$. By contradiction, lets assume that $\alpha \wedge (\alpha \wedge \neg ab_1 \rightarrow \mathbf{B}\beta) \not\models_{\sqsubseteq} \mathbf{B}\beta$. Then there is a model $M = \langle w_0, W\rangle$ in $\Gamma_1(\alpha \wedge (\alpha \wedge \neg ab_1 \rightarrow \mathbf{B}\beta))$ and $M \not\models \mathbf{B}\beta$, which means that there is a world $\overline{w} \in W$, such that $\overline{w} \not\models \beta$. We now have two cases, on whether $w_0$ satisfy or not $ab_1$.

- ## The proof is incomplete

  ∎

### 3.6 Strong Epistemic Cancellation holds for $\models_{\sqsubseteq}$

**Theorem 8 (SEC)** *If* $\alpha \models_{\leq} \beta$ *and* $\alpha \wedge \delta \models_{\leq} \gamma$, *then* $\mathbf{B}\alpha \wedge \mathbf{P}\delta \models_{\sqsubseteq} \mathbf{B}[(\beta \wedge \neg\delta) \vee (\gamma \wedge \delta)]$.

**Proof 8.** By contradiction, let us suppose that the conclusion is false, that is $\mathbf{B}\alpha \wedge \mathbf{P}\delta \not\models_{\sqsubseteq} \mathbf{B}[(\beta \wedge \neg\delta) \vee (\gamma \wedge \delta)]$. Therefore, there exists $\overline{M} = \langle w_0, \overline{W} \rangle \in \Gamma_1(\mathbf{B}\alpha \wedge \mathbf{P}\delta)$ and $\overline{M} \not\models \mathbf{B}[(\beta \wedge \neg\delta) \vee (\gamma \wedge \delta)]$.

This implies that there is $\overline{w} \in \overline{W}$ such that $\overline{w} \not\models \beta \wedge \neg\delta$ and $\overline{w} \not\models \gamma \wedge \delta$. Also since $\overline{M} \in \Gamma_1(\mathbf{B}\alpha \wedge \mathbf{P}\delta)$ then $\overline{w} \models \alpha$ and possibly $\overline{w} \models \delta$. Let us discuss both cases.

- **Case 1** $\overline{w} \models \delta$. Then, because $\overline{w} \not\models \gamma \wedge \delta$ one concludes that $\overline{w} \not\models \gamma$. This implies that $\overline{w} \notin \mu_1(\alpha \wedge \delta)$. Since $\overline{w} \in \mu_0(\alpha \wedge \delta)$, then there exists $\overline{\overline{w}} \in \mu_1(\alpha \wedge \delta)$ such that $\overline{\overline{w}} < \overline{w}$. Then the model $\overline{\overline{M}} = \langle w_0, \overline{W} - \{\overline{w}\} \cup \{\overline{\overline{w}}\} \rangle \sqsubset \overline{M}$, and since $\overline{\overline{M}} \in \Gamma_0(know\alpha \wedge \mathbf{P}\delta)$, we get a contradiction that $\overline{M}$ is minimal.
- **Case 2** $\overline{w} \not\models \delta$. Then, because $\overline{w} \not\models \beta \wedge \neg\delta$ one concludes that $\overline{w} \not\models \beta$, which in turn implies that $\overline{w} \notin \mu_1(\alpha)$. Therefore there exists $\overline{\overline{w}} < \overline{w}$. The model $\overline{\overline{M}} = \langle w_0, \overline{W} - \{\overline{w}\} \cup \{\overline{\overline{w}}\} \rangle \sqsubset \overline{M}$, is smaller (in the $\sqsubset$ sense) then $\overline{M}$, which contradicts with the claim that $\overline{M}$ is minimal.

∎

# 4 Conclusions

The author hopes this paper makes two important contributions. The first one is that it discusses some of the requirements that an epistemic nonmonotonic logic should meet. Although we developed these requirements based on a theory of utterance understanding, we believe that they are general requirements and should be used to compare different proposals of epistemic nonmonotonic logics.

The second contribution is the epistemic extension of preference logics. We discussed a method of extending any propositional preference logic, and proved that the resulting logics satisfy all requirements.

The research reported here is being expanded in two directions. The first one is the development of the epistemic extension of other nonmonotonic logics. We are currently developing the epistemic extension of conditional logics (for example [Bou92]). The second area of future research is the study of the requirements for multi-modal epistemic logics.

# References


[Bou92] Craig Boutilier. Conditional logics for default reasoning and belief revision. Technical Report KRR–TR–92–1, University of Toronto, Computer Science Department, 1992.

[Gri75] H. P. Grice. Logic and conversation. In P. Cole and J. L. Morgan, editors, *Syntax and Semantics 3: Speech Acts.* Academic Press, New York, 1975.

[HM85] J. Y. Halpern and Y. O. Moses. A guide to the modal logics of knowledge and belief: Preliminary draft. In *Proceedings of the 9th IJCAI*, pages 480–490, 1985.

[Lev90] Hector J. Levesque. All I know: A study in autoepistemic logic. *Artificial Intelligence*, 42:263–310, 1990.

[Lif85] Vladimir Lifschitz. Computing circumscription. In *Proc. of the Ninth International Joint Conference on Artificial Intelligence*, pages 121–127. Morgan Kaufman, 1985.

[McA88] Gregory L. McArthur. Reasoning about knowledge and belief: a survey. *Computational Intelligence*, 4:223–243, 1988.

[McC86] John McCarthy. Applications of circumscription to formalizing common-sense reasoning. *Artificial Intelligence*, 28:89–116, 1986.

[MWC91] A. Maida, J. Wainer, and S. Cho. A syntactic approach to introspection and reasoning about the beliefs of other agents. *Fundamenta Informaticae*, 15(3 and 4):333–356, 1991. Special Issue on Logics for Artificial Intelligence.

[Per90] C.. Raymond Perrault. An application of default logic to speech act theory. In P. R. Cohen, J. Morgan, and M. E. Pollack, editors, *Intentions in Communication*. The MIT Press, 1990.

[Rei80] Ray Reiter. A logic for default reasoning. *Artificial Intelligence*, 13:81–132, 1980.

[Sho87] Yoav Shoham. A semantical approach to non-monotonic logics. In *Proc. of the Tenth International Joint Conference on Artificial Intelligence*, pages 388–392, 1987.

[Sta92] Robert Stalnaker. What is a nonmonotonic consequence relation. In *Working Notes of the 4th International Workshop on Nonmonotonic Reasoning*, pages 218–230. Etherington and Kautz, Chairs, 1992.

[Wai92a] Jacques Wainer. Epistemic extension of preference logics. Technical Report CU–CS–619–92, University of Colorado, Department of Computer Science, Boulder, CO 80309, 1992.

[Wai92b] Jacques Wainer. Extending circumscription into modal domains. In *Proceedings of the 10th Meeting of the AAAI*, 1992.