# A Trust Region Algorithm For Nonlinearly Constrained Optimization

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**Abstract**

We present a trust region-based method for the general nonlinearly equality constrained optimization problem. The method works by iteratively minimizing a quadratic model of the Lagrangian subject to a possibly relaxed linearization of the problem constraints and a trust region constraint. The model minimization may be done approximately with a dogleg type approach. We show that this method is globally convergent even if singular or indefinite Hessian approximations are made.

A second order correction step that brings the iterates closer to the feasible set is described. If sufficiently precise Hessian information is used, this correction step allows us to prove that the method is also locally quadratically convergent, and that the limit satisfies the second order necessary conditions for constrained optimization. An example is given to show that, without this correction, a situation similar to the Maratos effect may occur where the iteration is unable to move away from a saddle point.

## 1. Introduction.

Algorithms involving trust regions have, over the past few years, proven to be effective and robust for solving unconstrained minimization problems. In this paper we describe a method using trust regions to solve the general nonlinearly equality constrained optimization problem

$$\underset{x \in R^n}{\text{minimize}} \ f(x) \tag{1.1}$$
$$\text{subject to} \ \ c(x) = 0.$$

Our method involves successive solution of a quadratic programming subproblem with an additional trust region constraint.

Methods based on successive quadratic programming generally involve iteratively minimizing a quadratic model of the Lagrangian subject to linear approximations of the constraints. That is, at $x_k$ they solve

$$\underset{d \in R^n}{\text{minimize}} \ g_k^T d + ½ d^T B_k d$$
$$\text{subject to} \ \ A_k^T d + c(x_k) = 0 \ ,$$

where $B_k$ is an approximation to the Hessian of the Lagrangian of problem (1.1), and set $x_{k+1} = x_k + d_k$ if this point is in some sense a better approximate solution to problem (1.1). Much analysis and experimentation has recently been done on successive quadratic programming methods involving a line search. However, for most versions of this approach only rather restrictive guarantees of global convergence can be given, and difficulties do occur in practice.

In unconstrained optimization use of a trust region has made it possible to make stronger guarantees of local convergence than can be made for a line search method. In particular, to guarantee that the gradient of the objective converges to zero, it is not necessary to require that the Hessian approximation be well-conditioned or even positive definite, but only that it be uniformly bounded Additionally, it can be shown that any cluster point of the algorithm has a positive semidefinite Hessian ( see Moré and Sorensen [1983] or Shultz, Schnabel, and Byrd [1985] ).

In this paper we describe and analyze a trust region method for problem (1.1). We show that it has the following properties which we regard as essential for a trust region method.

(1) If $\|B_k\|$ remains bounded above then the gradient of the Lagrangian and the constraint values converge to zero.

(2) If $B_k$ is a good approximation to the Hessian of the Lagrangian then the method is locally and quadratically convergent to a strong local minimizer, i.e. the Maratos effect does not occur.

(3) Under the same assumptions, if the algorithm converges to any point then that point satisfies the second order necessary conditions for a local optimum.

(4) The trust region step can be computed efficiently.

The only assumptions we make are that the derivatives are bounded, that the merit function is bounded below on the sequence of iterates, and that the constraint derivatives at the iterates are uniformly linearly independent. Although this last assumption is a common one, we are not satisfied with it, and we are currently working on making the algorithm reliable in the presence of linearly dependent constraint derivatives.

There has been some previous work on trust region algorithms for nonlinear constraints, and this paper has some features in common with this work. Our strategy for relaxing constraints is similar to that of Vardi [1985], but is designed to weaken the conditions put on $B_k$ corresponding to property (1) above. Our goals are quite similar to those of the algorithm proposed by Fletcher [1982], [1984], even though that algorithm involves minimization of the merit function over a box shaped trust region. Such a local minimization involves solution of a quadratic program even if there are only equality constraints. If the Hessian approximation is indefinite then the quadratic program may even have several isolated local minima. Our approach also has some similarities to the method described by Celis, Dennis, and Tapia [1984] which is currently under development.

We believe that this method and its analysis are of interest because we are able to prove the first order global convergence property (1) while computing a trust region step in a manner analogous to and as inexpensive as step computation in unconstrained optimization. We also believe that this is the first time that second order global convergence (property (2) ) has been proved for a nonlinearly constrained optimization algorithm. Proving this is complicated by the fact that a phenomenon similar to the Maratos effect can occur involving directions of negative curvature of the Lagrangian . In Section 4 we describe how this problem can be avoided by using a second order correction.

Section 2 of this paper gives a motivation and description of the algorithm, and Section 3 contains the first order global convergence analysis. Section 4 has an example illustrating the need for a second order correction, and proves second order global convergence and local convergence.

## 2. Description of the Algorithm.

We first introduce some standard notation for the paper.

### Notation

Let $|| \ ||$ be the Euclidean norm on $R^n$.

Let $f : R^n \to R$ be twice continuously differentiable, with gradient $g : R^n \to R$ and Hessian matrix $\nabla^2 f$.

Let $c : R^n \rightarrow R^m$ be the vector of twice continuously differentiable constraint functions $c^i(x)$, for $i = 1,...,m$, with the gradient of $c^i(x)$ denoted by $a^i(x)$, and the Hessian matrix of $c^i(x)$ denoted by $\nabla^2 c^i(x)$.

Let $A(x)$ be the $n$ by $m$ matrix consisting of the column vectors $a^i(x)$, for $i = 1,...,m$. For any $x$, let $Z(x)$ be an $n$ by $n-m$ matrix whose columns form an orthonormal basis for the null space of A ofx sup T, i.e. such that $A(x)^T Z(x) = 0$ and $Z(x)^T Z(x) = I$.

Denote the first order Lagrange multiplier estimates by

$$\lambda(x) = -[A(x)^T A(x)]^{-1} A(x)^T g(x) .$$

Denote the least-squares step on the constraints by

$$v(x) = -A(x)[A(x)^T A(x)]^{-1} c(x) .$$

Let $\nu_1(H)$ be the smallest eigenvalue of an $n$ by $n$ symmetric matrix $H$.

Let $\{x_k\}$ be a sequence of iterates generated by the algorithm, and for each $k$ let $B_k$ be an $n$ by $n$ symmetric matrix.

Subscripted values of functions denote evaluation at a point in the sequence of iterates, while superscripts denote the particular component of a vector. For example, $g_k = g(x_k)$, and $a_k^i$ is the $i$-th column of $A(x_k)$.

We will frequently delete subscripts, superscripts, and function arguments when they are clear from the context.

The algorithm we present here makes use of a trust region of radius $\Delta$ within which a quadratic model of the Lagrangian function and a linear model of the constraints are believed to be accurate.

At each iterate a quadratic model of the Lagrangian function in a neighborhood of $x_k$

$$q(x_k) = g_k^T d + \tfrac{1}{2} d^T B_k d \qquad (2.1)$$

is approximately minimized subject to a relaxed linearized version of the constraints

$$A_k^T d = -\alpha c(x_k) . \tag{2.2}$$

and a trust region constraint

$$\| d \| \leq \Delta . \tag{2.3}$$

Using a relaxed version of the linearized constraints (i.e. allowing $\alpha < 1$ in (2.2)) allows us to define a step when the linearized feasible region does not intersect the trust region, and it gives us the freedom to work on reducing the objective function as well as the constraints.

The relaxation factor $\alpha$ can be at most one and must be chosen so that the intersection of the sets given by (2.2) and (2.3) is not empty and optionally so that it contains more than one point. To motivate the choice of $\alpha$, note that any solution to (2.1-2.3) must have the form

$$d = \alpha v_k + Z_k u. \tag{2.4}$$

where $Z_k$ is as defined above,

$$v_k = v(x_k) = -A_k(A_k^T A_k)^{-1} c(x_k),$$

and u is a vector in $R^{n-m}$. Thus $\alpha v_k$ is the component of $d$ in the range space of $A_k$, and $Z_k u$ is the component in the null space of $A_k^T$. In these terms our trust region constraint requires that

$$\alpha \| v_k \| \leq \Delta .$$

In addition, to ensure that significant progress is made on satisfying the constraints we require that, when $\alpha < 1$, the constraint range space step not be too small relative to the the trust region, i.e. that

$$\alpha \| v_k \| \geq \theta \Delta$$

where $\theta$ is a constant. Putting these conditions together results in an $\alpha$ interval given in stage (5) of the algorithm description below.

The null space component $Z_k u$ is determined by minimizing the quadratic model (2.1). In terms of the decomposition (2.4) our problem becomes

$$\text{minimize } (g_k + B_k \alpha v_k)^T Z_k u + \tfrac{1}{2} u^T Z_k^T B_k Z_k u \tag{2.5}$$
$$\text{subject to } \|u\|^2 \le \Delta^2 - (\alpha \|v_k\|)^2$$

Note that this has the same form as a trust region step in an unconstrained algorithm. Thus if we minimize the model exactly the solution is usually given by

$$u_k = -(Z_k^T B_k Z_k + \beta I)^{-1} Z_k^T (g_k + B_k \alpha v_k)$$

where $\beta$ can be chosen to satisfy the trust region constraint and complementary slackness. Alternatively, the minimization can be done approximately, for example by a dogleg technique, or by any of the approaches described in the paper by Shultz, Schnabel, and Byrd [1985]. We will discuss later the conditions that the approximate minimization of (2.5) must satisfy. Note that such a step is well defined and reasonable even if the matrix $B_k$ is singular or indefinite.

Now, given a step $d_k = \alpha v_k + Z_k u_k$ that satisfies (2.1-2.3), we test it to determine whether the suggested point $x_k + d_k$ has improved on the objective function and constraints. We will measure improvement by the merit function

$$\phi(x) = f(x) + \sum_{i=1}^{m} \mu^i |c^i(x)|$$

where the $\mu^i$ are positive weights. As is pointed out by Coleman and Conn [1980], this function has the advantage that any local minimum of (1.1) is a stationary point of $\phi$. The actual reduction in the merit function in going from $x_k$ to $x_{k+1}$ is thus given by

$$ared_k(d_k) = f(x_k) - f(x_k + d_k) + \sum_{i=1}^{m} \mu^i \{|c^i(x_k)| - |c^i(x_k + d_k)|\},$$

where $d_k$ is the step computed by the algorithm at $x_k$. This step $d_k$ is based on approximations to the objective function and constraints. Using these same approximations we can compute a prediction of what this reduction will be according to our model:

$$pred_k(d_k) = -g_k^T d_k - \tfrac{1}{2} d_k^T B_k d_k + \sum_{i=1}^{m} \mu^i \{ |c^i(x_k)| - |c^i(x_k) + a^i(x_k)^T d_k| \} .$$

If the improvement in $\phi$ is a sufficient proportion of that predicted by our model, i.e. if

$$\frac{ared}{pred} \geq \eta,$$

where $\eta < 1$ is a fixed constant, then the step is accepted and $x_{k+1} = x_k + d_k$. Otherwise, we reduce the radius of the trust region and compute a new provisional step $d_k$ which is again compared to $\phi$.

It is of course essential that if the approximations used in generating the step are good, then our merit function will recognize the step as an improvement. This means that the prediction of the merit function reduction, $pred_k$, must always indicate a reduction unless $x_k$ is optimal. It turns out that that property is guaranteed if the weights are sufficiently large. In particular, the predicted decrease is at least as great as the approximate change to the Lagrangian due to the step $Z_k u$ defined by (2.5), plus a term proportional to the norm of the constraints. The former quantity is denoted by

$$hpred_k(u_k) = -\tilde{g}_k^T Z_k u_k - \tfrac{1}{2} u_k^T Z_k^T B_k Z_k u_k ,$$

where

$$\tilde{g}_k = g_k + \alpha_k B_k v_k .$$

Our assumption on the size of the weights involves the quantities

$$\tilde{\mu}_k = -(A_k^T A_k)^{-1} A_k^T (g_k + \tfrac{1}{2}\alpha_k B_k v_k) .$$

These multiplier-like quantities $\tilde{\mu}$ express the balance between change in the objective function and change in the constraints in that

$$q(0) - q(\alpha v) = -\alpha v^T \tilde{g} = -\alpha c^T \tilde{\mu} .$$

Note that similar assumptions on the weights are required in algorithms doing a line search on the merit function (see for example Han [1977]).

**Lemma 1**

Let $d_k$ be a step generated by solving (2.1-2.3) at an iterate $x_k$. Assume that $A_k$ has full rank and $\mu^i \geq \tilde{\mu}_k^i + \rho$, where $\rho > 0$ is a fixed constant. Then,

$$pred_k(d_k) \geq hpred_k(u_k) + \rho\alpha_k \|c_k\|_1 .$$

**Proof:**

To simplify the notation, we omit the subscripts $k$. Note that

$$\begin{aligned} pred &= -g^T(\alpha v + Zu) - \tfrac{1}{2}(\alpha v + Zu)^T B(\alpha v + Zu) \\ &\quad + \sum_{i=1}^{m} \mu^i\{|c^i| - |c^i + a^{iT}(\alpha v + Zu)|\} \\ &= -u^T Z^T(g + \alpha Bv) - \tfrac{1}{2}u^T Z^T BZu - \alpha v^T(g + \tfrac{1}{2}\alpha Bv) + \alpha \sum_{i=1}^{m} \mu^i |c^i| , \end{aligned}$$

by rearranging the first two terms and using the fact that $a^{iT}(\alpha v + Zu) = -\alpha c^i$. By assumption, $\mu^i \geq \tilde{\mu}^i + \rho$, where $\tilde{\mu} = -(A^TA)^{-1}A^T(g + \tfrac{1}{2}\alpha Bv)$, and $v = -A(A^TA)^{-1}c$, hence

$$v^T(g + \tfrac{1}{2}\alpha Bv) = \tilde{\mu}^T c = \sum_{i=1}^{m} \tilde{\mu}^i c^i ,$$

so

$$\begin{aligned} -\alpha v^T(g + \tfrac{1}{2}\alpha Bv) + \alpha \sum_{i=1}^{m} \mu^i |c^i| &= -\alpha \sum_{i=1}^{m} \tilde{\mu}^i c^i + \alpha \sum_{i=1}^{m} \mu^i |c^i| \\ &\geq \alpha\rho \|c\|_1 . \end{aligned}$$

□

The features described above are sufficient, with a few minor details, to guarantee first order global convergence in the sense that, under reasonable conditions, any limit point satisfies the Kuhn-Tucker conditions. However, just as with line search methods, use of a nondifferentiable merit function can cause difficulties by requiring the iterates to stay closer to the feasible set than the generated step naturally tends to fall. Two of these difficulties involve convergence theory; one is well known, the other less so. The first of these, sometimes referred to as the Maratos effect, has been noted by Maratos [1978], and, in a trust region context, by Yuan [1984]. If the weights $\mu^i$ are too large, a step which moves closer to the solution, makes progress on the objective, and keeps the constraints reasonably small can actually increase the merit function even near a solution, and superlinear convergence will not occur.

We show in section 4 that a related phenomenon can occur when we try to follow a direction of negative curvature of the Lagrangian. Indeed, it can happen that no step along a direction of negative curvature will decrease the merit function.

One way to get around these problems is to add a correction step to $d_k$ that moves closer to the feasible region. In the context of the Maratos effect, such steps have been suggested by Mayne and Polak [1982], by Coleman and Conn [1982], and by Fletcher [1982], [1984]. Our algorithm takes in stage (12) a step of the form

$$-A_k(A_k^T A_k)^{-1} c(x_k + d_k)$$

in cases where the merit function increases and $v_k$ is small relative to the trust region ( $\|v\| \leq \zeta\Delta$). Note that the condition $\zeta \epsilon (0,\theta)$, implies that the correction is made only if the linearized constraints are not relaxed. In Section 4 we show how this takes care of both difficulties.

We now give the formal description of our algorithm. Note that in stage (7) the null-space component $u$ of the step is assumed to satisfy Condition #1 and perhaps Condition #2 and #3. These Conditions on the null-space component of the step are given following the description of the algorithm.

**Algorithm**

Given $\eta \epsilon (0,1)$, $\theta \epsilon (0,1]$, $\zeta \epsilon (0,\theta)$, $\tau_1$, $\tau_2$ $\epsilon (0,1)$, and $\mu \epsilon R^m$, with $\mu > 0$.

(0) $k=0$; input $x_0$.

(1) Compute $x = x_k$, $f = f(x)$, $g = g(x)$, $c = c(x)$,

$A = A(x)$, $Z = Z(x)$, and pick $B \epsilon R^{n \times n}$.

(2) If $Z^T g = 0$ and $c = 0$ then stop.

(3) Pick initial $\Delta \geq \Delta_{k-1}$, with any $\Delta_0 > 0$ if $k = 0$.

(4) Compute $v = -A(A^TA)^{-1}c$.

(5) Pick $\alpha \epsilon [\min\{1, \theta \frac{\Delta}{\|v\|}\}, \min\{1, \frac{\Delta}{\|v\|}\}]$.

(6) Compute $\tilde{\Delta} = (\Delta^2 - \alpha^2 \|v\|^2)^{1/2}$.

(7) Compute the null-space step $u$ satisfying Condition #1 and optionally satisfying Conditions #2 and #3 (these conditions are described below), and satisfying $\|u\| \leq \tilde{\Delta}$; let $d = \alpha v + Zu$.

(8) Compute $pred = -g^Td - \frac{1}{2} d^T B d + \sum_{i=1}^{m} \mu^i \{|c^i| - |c^i + a^{iT} d|\}$ .

(9) Compute $f_+ = f(x+d)$ and $c_+ = c(x+d)$.

(10) Compute $ared = f - f_+ + \sum_{i=1}^{m} \mu^i \{|c^i| - |c^i_+|\}$ .

(11) if $\frac{ared}{pred} \geq \eta$ then $x_{k+1} = x + d$; $k = k+1$; and go to (1)

else if $\|v\| \leq \zeta \Delta$ then $ared = f - f(x+d+w) + \sum_{i=1}^{m} \mu^i \{|c^i| - |c^i(x+d+w)|\}$,

where $w = -A(A^TA)^{-1} c_+$ ,

if $\frac{ared}{pred} \geq \eta$

then $x_{k+1} = x + d + w$; $k=k+1$; and go to (1)

let $\Delta = \tau \Delta$ for some $\tau \epsilon [\tau_1, \tau_2]$ and go to (5).

**Comments**

The condition, $\tau \epsilon [\tau_1, \tau_2]$, in the last step allows a variety of trust region modification strategies, for example safeguarded interpolation.

We now give our theoretical results.

## 3. First Order Global Convergence.

For the global and local convergence results to follow, we will require certain continuity and boundedness assumptions about the problem being solved.

**Standard Assumptions**

Let $\{x_k\}$ be generated by the algorithm with any initial iterate $x_0 \in R^n$ and suppose that $\{x_k\}$ is contained in some open subset $S$ of $R^n$. Assume that $f(x)$ and $c(x)$ are twice continuously differentiable on $S$, and assume that $A(x)$, $g(x)$, $\nabla^2 f(x)$, $(A(x)^T A(x))^{-1}$, $\nabla^2 f(x)$, and each $\nabla^2 c^i(x)$, for $i = 1,\ldots,m$, are all bounded on $S$. Assume further that for some $\rho > 0$, $\mu^i \geq \tilde{\mu}_k^i + \rho$ for all $k$, and that for some $\beta > 0$, $\|B_k\| \leq \beta$ for all $k$.

An immediate consequence of these assumptions is that there is some $\gamma_1 > 0$ such that

$$\|v(x)\| \leq \gamma_1 \ \|c(x)\|_1$$

for all $x \in S$.

Note that if the sequence of iterates were contained in some compact set the assumptions on boundedness of various derivatives would follow from continuity.

The next lemma shows that our predicted reduction of the merit function provides an approximation to the merit function that is accurate to within the square of the steplength. Note that this result does not depend on any property of the matrices approximating the Hessian of the Lagrangian except that they remain bounded, and does not depend on any property of the step.

**Lemma 2.**

Suppose that the Standard Assumptions hold. Then there is an $\gamma_2 > 0$ such that for any $x \in S$, $B \in R^{n \times n}$ with $\|B\| \leq \beta$, and any $d \in R^n$, with the line segment from $x$ to $x+d$ contained in $S$,

$$|ared(d) - pred(d)| \leq \gamma_2 \|d\|^2 ,$$

where

$$pred(d) = -g(x)^T d - ½ d^T B d + \sum_{i=1}^{m} \mu^i \{|c^i(x)| - |c^i(x) + a^i(x)^T d|\}$$

and

$$ared(d) = f(x) - f(x+d) + \sum_{i=1}^{m} \mu^i \{|c^i(x)| - |c^i(x+d)|\} .$$

**Proof:**

Consider any $x \in S$, $d \in R^n$, with the line segment from $x$ to $x+d$ contained in $S$, and $B \in R^{n \times n}$ with $\|B\| \leq \beta$. Then by the definition of *ared* and *pred*,

$$|ared - pred| = |\{f - f(x+d) + \sum_{i=1}^{m} \mu^i (|c^i| - |c^i(x+d)|)\}$$

$$- \{-g^T d - ½ d^T B d + \sum_{i=1}^{m} \mu^i (|c^i| - |c^i + a^{iT} d|)\} |$$

$$\leq |f - (f + g(\xi)^T d) + g^T d + ½ d^T B d|$$

$$+ | \sum_{i=1}^{m} \mu^i \{ |c^i + a^{iT} d| - |c^i + a^i(\xi_i)^T d| \} |$$

$$\leq \|d\| \|g(x) - g(\xi)\| + ½ \|B\| \|d\|^2 + \sum_{i=1}^{m} \mu^i \|d\| \|a^i(x) - a^i(\xi_i)\| ,$$

for some $\xi$, $\xi_i$ on the line segment between $x$ and $x+d$, from the mean value theorem. Then since $\nabla^2 f$ and $\nabla^2 c^i$, for $i = 1,\ldots,m$, are bounded on $S$, and $\|B\| \leq \beta$, the result follows. □

The next result shows that the algorithm is well-defined in the sense that each inner iteration will terminate with an acceptable step after finitely many iterations. It is clear from the proof of this theorem that the same result holds irrespective of the strategy used for deciding when to attempt the second order correction step.

**Theorem 1**

Suppose that the Standard Assumptions hold and that the null-space components $u$ in stage (7) of the algorithm satisfy Condition #1. Then unless some iterate $x_k$ satisfies the first order necessary conditions for a solution to (1.1), each inner iteration of the algorithm will terminate after finitely many repetitions.

**Proof:**

Consider any iterate $x_k$. As usual, we omit the subscripts. Consider first the test $\frac{ared(d)}{pred} \geq \eta$ which is made at each inner iteration with decreasing values of $\Delta$.

Suppose first that $\|c\|_1 > 0$, and consider any $\Delta > 0$. From the algorithm, stage (5), $\alpha \geq \min\{1, \theta \frac{\Delta}{\|v\|}\}$. By Lemma 1 and since $hpred(u) \geq 0$, $pred(d) \geq \alpha\rho \|c\|_1$, so

$$pred(d) \geq \rho \min\{\|c\|_1, \theta\Delta \frac{\|c\|_1}{\|v\|}\} .$$

But, since $\|v(x)\| \leq \gamma_1 \|c(x)\|_1$ for all $x \in S$,

$$pred(d) \geq \rho \min\{\|c\|_1, \frac{\theta}{\gamma_1}\Delta\} .$$

Thus, since by Lemma 2

$$|ared(d) - pred(d)| \leq \gamma_2 \|d\|^2 \leq \gamma_2 \Delta^2$$

for some $\gamma_2 > 0$, it follows that for all small enough $\Delta > 0$, $|\frac{ared(d)}{pred(d)} - 1| \leq 1 - \eta$, so after finitely many repetitions, the step will be accepted.

On the other hand, if $c = 0$, then $v = 0$, so $\tilde{g} = g$ and $\tilde{\Delta} = \Delta$, and by Condition #1 and Lemma 1,

$$pred(d) \geq \kappa_1 \|Z^T g\| \min\{\Delta, \frac{\|Z^T g\|}{\beta}\} .$$

Hence, if $\|Z^T g\| > 0$, since by Lemma 2, $|ared(d) - pred(d)| \leq \gamma_2 \Delta^2$, it follows that for all small enough $\Delta > 0$, the step will be accepted.

Of course, it can happen that before $\Delta$ is that small, the step $d + w$ is tested and accepted; in this case the iteration is still finite. □

We now give the first order global convergence result, that the constraint violations and the projected gradients converge to 0. Hence, any accumulation point of the sequence of iterates satisfies the first order necessary conditions for a solution to (1.1).

**Theorem 2**

Let the Standard Assumptions hold. Assume that $\{\phi(x_k)\}$ is bounded below on $S$, and that the null-space components $u$ in stage (7) of the algorithm satisfy Condition #1. Then

a) $c_k \to 0$ and b) $Z_k^T g_k \to 0$.

**Proof:**

Suppose to the contrary that there is some $\epsilon > 0$ such that there are infinitely many $k$ with either $\|c_k\|_1 \geq \epsilon$ or $\|Z_k^T g_k\| \geq \epsilon$. From the Standard Assumptions it is clear that $\|c(x)\|$ and $\|Z(x)^T g(x)\|$ are uniformly continuous on $S$. Thus, there is some $r > 0$ such that for any $x$, if $\|c_k\|_1 \geq \epsilon$ and $\|x - x_k\| < r$, then $\|c(x)\|_1 \geq \frac{1}{2}\epsilon$, and if $\|Z_k^T g_k\| \geq \epsilon$ and $\|x - x_k\| < r$, then $\|Z(x)^T g(x)\| \geq \frac{1}{2}\epsilon$.

Consider an $x_k$ with $\|c_k\|_1 \geq \epsilon$. By Lemma 1, for any $x$ with $\|x - x_k\| < r$, and any $\Delta > 0$,

$$pred(d) \geq \rho\alpha \|c(x)\|_1 \geq \rho \min\{\frac{\epsilon}{2}, \theta\Delta \frac{\|c\|_1}{\|v\|}\}$$

$$\geq \min\{\rho\frac{\epsilon}{2},\rho\frac{\theta}{\gamma_1}\Delta\},$$

since $\|v\| \leq \gamma_1 \|c\|_1$.

Now, consider an iterate $x_k$ with $\|Z_k^T g_k\| \geq \epsilon$, and any $x$ with $\|x-x_k\| < r$. First consider the case that $\|v\| \leq \min\{\frac{\epsilon}{4\beta},\zeta\Delta\}$. Then by Lemma 1 and Condition #1,

$$pred(d) \geq \kappa_1 \|Z^T\tilde{g}\| \min\{\tilde{\Delta},\frac{\|Z^T\tilde{g}\|}{\|Z^TBZ\|}\}.$$

Since $\tilde{\Delta}^2 = \Delta^2-\alpha^2\|v\|^2$, $\alpha \leq 1$, and $\|v\| \leq \zeta\Delta$, it follows that $\tilde{\Delta} \geq (1-\zeta^2)^{½}\Delta$. Also,

$$\|\alpha Z^TBv\| \leq \|B\|\,\|v\| \leq \beta\frac{\epsilon}{4\beta},$$

and $\|Z^Tg\| \geq \frac{\epsilon}{2}$, so

$$\|Z^T\tilde{g}\| = \|Z^T(g+\alpha Bv)\| \geq \frac{\epsilon}{4}.$$

Thus,

$$pred(d) \geq \kappa_1\frac{\epsilon}{4}\min\{(1-\zeta^2)^{½}\Delta,\frac{\epsilon}{4\beta}\}.$$

Second, suppose that $\|v\| > \min\{\frac{\epsilon}{4\beta},\zeta\Delta\}$. Then by Lemma 1,

$$pred(d) \geq \rho\alpha\|c\|_1 \geq \rho\min\{1,\theta\frac{\Delta}{\|v\|}\}\|c\|_1$$

$$\geq \rho\min\{\frac{\|v\|}{\gamma_1},\frac{\theta}{\gamma_1}\Delta\}$$

$$\geq \frac{\rho}{\gamma_1}\min\{\min\{\frac{\epsilon}{4\beta},\zeta\Delta\},\theta\Delta\}.$$

Thus we have shown that there are positive constants $\rho_1$, $\rho_2$, and $r$ such that for infinitely many $x_k$, and any $\Delta > 0$, with $\|d_j\| \leq \Delta$,

$$pred(d_j) \geq \min\{\rho_1,\rho_2\Delta\}$$

for all iterates $x_j$ within a distance $r$ of each such $x_k$. Thus, by Lemma 2, there is some $\overline{\Delta} > 0$ such that for all $k$ with $\|c_k\|_1 \geq \epsilon$ or $\|Z_k^Tg_k\| \geq \epsilon$, and for any

$\Delta \leq \bar{\Delta}$, if $\|x_j - x_k\| < r$ then the trust region step $d_j(\Delta)$ will be accepted, and will yield

$$ared_j(d_j) \geq \eta pred_j(d_j) \geq \eta \min\{\rho_1, \rho_2 \Delta\} .$$

Consider one such $x_k$. If all the iterates after $x_k$ stay in the ball of radius $r$ about $x_k$, then the trust region radius will be bounded away from 0, which contradicts the assumption that $\{\phi(x_k)\}$ is bounded below on $S$. So, for one such $x_k$, let $j \geq k$ be the smallest index such that $\|x_{j+1} - x_k\| \geq r$. Then if no steps with the second order correction are made from $x_k$ to $x_j$,

$$\phi(x_k) - \phi(x_{j+1}) = \sum_{i=k}^{j} ared_i(d_i) \geq \sum_{i=k}^{j} \eta \; pred_i(d_i)$$

$$\geq \eta \sum_{i=k}^{j} \min\{\rho_1, \rho_2 \Delta_i\}$$

$$\geq \eta \min\{\rho_1, r\} ,$$

since each $\|d_i(\Delta_i)\| \leq \Delta_i$ and $\sum_{i=k}^{j} \|d_i(\Delta_i)\| \geq r$. If, on the other hand, a second order correction step $d_i + w_i$, for some $i$, with $k \leq i \leq j$, is taken, then it must be that $\Delta_i > \bar{\Delta}$, so

$$\phi(x_k) - \phi(x_{j+1}) \geq ared_i(d_i + w_i)$$

$$\geq \eta pred_i(d_i)$$

$$\geq \eta \min\{\rho_1, \rho_2 \Delta_i\} \geq \eta \min\{\rho_1, \rho_2 \bar{\Delta}\} .$$

In either case, we have a contradiction of the assumption that $\{\phi(x_k)\}$ is bounded below on $S$, since there are infinitely many $x_k$ yielding this decrease in $\phi$. Hence, $c_k \to 0$ and $Z_k^T g_k \to 0$. □

Note that to prove this theorem the second order correction in Step (11) was not actually needed, and that the algorithm without such a correction is still globally convergent in the sense of this result. It is also worth noting that we did not assume in the statement of the theorem that the sequence of iterates was bounded. Of course, if the sequence were bounded it would have one or more cluster points, all of which would satisfy the Kuhn-Tucker conditions.

Theorem 2 and all the other theorems in this paper assume fixed weights $\mu^i$ satisfying $\mu^i \geq \tilde{\mu}_k^i + \rho$ for all $k$ and for some $\rho > 0$. In fact the weights may be chosen dynamically by requiring, for example, that $\mu^i$ at step $k$ be the maximum of $\tilde{\mu}_k^i + \rho$ and the $\mu^i$ at the previous step. Since, by the Standard Assumptions, $\tilde{\mu}_k^i$ are bounded for all $k$ this procedure would only increase $\mu$ finitely many times, after which the weights would be fixed.

## 4. Second Order Results.

Now we discuss the need for a second order correction step. As is well known, it can happen that in a neighborhood of a strong minimizer the step generated by successive quadratic programming may not decrease a nondifferentiable merit function such as $\phi$. In order to decrease the merit function, a very short step may be required, and this can impede superlinear convergence. As is shown implicitly by Coleman and Conn [1982] and by Mayne and Polak [1982], if a second order correction step of the form

$$w_k = -A_k[A_k^T A_k]^{-1} c(x_k + d_k)$$

is added to an SQP step $d_k$ then a line search algorithm can be made superlinearly convergent.

An additional reason for requiring a device such as a second order correction is involved when trying to move away from a point where the Hessian of the Lagrangian is indefinite. In this case our trust region algorithm will move along a direction of negative curvature. However, it can happen, as seen in the following example, that any step along a direction of negative curvature of the Lagrangian will increase the merit function.

**Example.**

$$\text{minimize } 2x_1 + \tfrac{1}{2}x_2^2$$

$$\text{subject to } x_1^2 + x_2^2 = 1.$$

The only local minimum is at (-1, 0), but there is a Kuhn-Tucker point $x = (1,0)$ with Lagrange multiplier $\lambda = -1$. The Hessian of the Lagrangian at that point is

$$\nabla^2 L(x,-1) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$Z^T \nabla^2 L(x,1) Z = \begin{bmatrix} -1 \end{bmatrix}.$$

Note that the direction $d = (0,1)^T$ is a direction of negative curvature satisfying $A(x)^T d = -c(x) = 0$. If our algorithm were exactly minimizing the quadratic model over a trust region of any radius $\Delta$, the step from $x$ would have the form $\Delta d$. However, a step of any length along the direction $d$ would increase both the objective function and the absolute value of the constraint. Therefore, for any radius $\Delta$ and any positive weight $\mu$ the trust region step will increase the merit function. □

This example does not seem especially pathological. The key ingredient seems to be that both constraint and objective have curvature in the same direction, and the constraint curvature is greater. The phenomenon is similar to the Maratos effect but is a bit more problematical in that problems occur regardless of the length of the step or of the weights. It is clear that to decrease the merit function it is necessary to follow the constraint curve more closely. In the following results we show that the second order correction step in the algorithm is sufficient to get around this difficulty as well as the Maratos effect. Note that in the algorithm the correction is not made at every step, but only when the trust region step $d_k$ does not decrease the merit function sufficiently, the constraints are not relaxed, and $\|v_k\|$ is at most a fixed fraction of $\Delta$. Thus the second order correction will probably be made only at iterates fairly close to the feasible set.

To begin with we prove the essential property of the second order correction step, namely that the actual reduction of the merit function obtained by use of the second order correction agrees with the predicted reduction of the trust region step $d_k$ to within $o(\|d_k\|^2)$.

**Lemma 3**

Let the Standard Assumptions hold, and assume that

$$B_k = \nabla^2 f(x_k) + \sum_{i=1}^{m} \lambda^i(x_k) \nabla^2 c^i(x_k) .$$

Then for any compact subset $D$ of $S$, for any $\epsilon > 0$, there is a $\bar{\Delta} > 0$ such that for any iterate $x_k \epsilon D$ at which the second order correction step $d_k + w_k$ is taken and $\|d_k\| \leq \bar{\Delta}$,

$$|ared_k(d_k + w_k) - pred_k(d_k)| \leq \epsilon \ \|d_k\|^2 .$$

**Proof:**

Consider any compact set $D$ contained in $S$, and any $\epsilon > 0$, and an iterate $x \epsilon D$ at which the second order correction step $d + w$ as in stage (11) of the algorithm is taken, where as usual we drop the subscripts $k$.

By definition,

$$\begin{aligned} |ared(d+w) - pred(d)| &= |\{f - f(x+d+w) + \sum_{i=1}^{m} \mu^i(|c^i| - |c^i(x+d+w)|)\} \\ &\quad - \{-g^T d - \tfrac{1}{2} d^T B d + \sum_{i=1}^{m} \mu^i(|c^i| - |c^i + a^{iT} d|)\}| \\ &\leq |f + g^T d + \tfrac{1}{2} d^T B d - f(x+d+w)| \\ &\quad + |\sum_{i=1}^{m} \mu^i \{|c^i + a^{iT} d| - |c^i(x+d+w)|\}| . \end{aligned}$$

Since the second order correction is only attempted if $\|v\| \leq \zeta\Delta$ and since $\zeta \epsilon (0, \theta)$, it follows that $\alpha = 1$ and $c^i + a^{iT} d = 0$.

We first show that the constraint part is small enough. For each $i$,

$$c^i(x+d) = c^i(x) + a^i(x)^T d + \tfrac{1}{2} d^T \nabla^2 c^i(\xi_i) d$$
$$= \tfrac{1}{2} d^T \nabla^2 c^i(\xi_i) d$$

for some $\xi_i$ on the line segment between $x$ and $x+d$. Since $w = -A(A^TA)^{-1}c(x+d)$, and each $\nabla^2 c^i$ is bounded, there is a constant $\gamma_7$ such that $\|w\| \le \gamma_7 \|d\|^2$. Also,

$$c^i(x+d+w) = c^i(x+d) + a^i(x)^T w + (a^i(\overline{\xi}_i) - a^i(x))^T w$$
$$= (a^i(\overline{\xi}_i) - a^i(x))^T w$$

for some $\overline{\xi}_i$ on the line segment between $x$ and $x+d+w$. So, there is a constant $\gamma_5$ such that

$$|c^i(x+d+w)| \le \|a^i(\overline{\xi}_i) - a^i(x)\| \, \|w\| \le \gamma_5 \|d\|^3 .$$

Thus,

$$\left| \sum_{i=1}^{m} \mu^i \{ |c^i(x+d+w)| - |c^i + a^{iT} d| \} \right| \le \gamma_6 \|d\|^3 ,$$

for some constant $\gamma_6$.

Next, to show that the quadratic model of the Lagrangian is close enough to $f(x+d+w)$, note that

$$| f + g^T d + \tfrac{1}{2} d^T B d - f(x+d+w) |$$
$$= |f + g^T d + \tfrac{1}{2} d^T \nabla^2 f(x) d - f(x+d+w)$$
$$+ \tfrac{1}{2} \sum_{i=1}^{m} \lambda^i d^T \nabla^2 c^i d |$$
$$= |f + g^T d + \tfrac{1}{2} d^T \nabla^2 f(x) d - f(x+d) - g(x+d)^T w - \tfrac{1}{2} w^T \nabla^2 f(\overline{\xi}) w$$
$$+ \tfrac{1}{2} \sum_{i=1}^{m} \lambda^i d^T \nabla^2 c^i d |$$
$$= |f + g^T d + \tfrac{1}{2} d^T \nabla^2 f(x) d - f - g^T d - \tfrac{1}{2} d^T \nabla^2 f(\xi) d - g(x+d)^T w - \tfrac{1}{2} w^T \nabla^2 f(\overline{\xi}) w$$
$$+ \tfrac{1}{2} \sum_{i=1}^{m} \lambda^i d^T \nabla^2 c^i d |$$
$$\le \tfrac{1}{2} | d^T (\nabla^2 f(x) - \nabla^2 f(\xi)) d | + \tfrac{1}{2} | w^T \nabla^2 f(\overline{\xi}) w |$$

$$+ \ \left| -g(x+d)^T w + \tfrac{1}{2} \sum_{i=1}^{m} \lambda^i d^T \nabla^2 c^i d \right|$$

for some $\bar{\xi}$ on the line segment between $x+d$ and $x+d+w$ and $\xi$ on the line segment between $x$ and $x+d$. The first two terms will clearly be smaller than any constant multiplied by $\|d\|^2$ for all small enough $d$, since $\nabla^2 f$ is uniformly continuous on $D$ and $\|w\| \le \gamma_7 \|d\|^2$.

For the last term, note that since

$$g(x+d)^T w = \bar{\lambda}^T c(x+d) ,$$

where $\bar{\lambda} = -(A^TA)^{-1}A^T g(x+d)$, we have that

$$\left| -g(x+d)^T w + \sum_{i=1}^{m} \tfrac{1}{2} \lambda^i d^T \nabla^2 c^i d \right|$$

$$\le \left| \sum_{i=1}^{m} \bar{\lambda}^i (c^i + a^{iT} d + \tfrac{1}{2} d^T \nabla^2 c^i(\xi_i) d - \tfrac{1}{2} d^T \nabla^2 c^i(x) d) \right|$$

$$+ \left| \sum_{i=1}^{m} (\lambda^i - \bar{\lambda}^i) \tfrac{1}{2} d^T \nabla^2 c^i(x) d \right|$$

$$\le \sum_{i=1}^{m} |\bar{\lambda}^i| \, \|d\|^2 \, \|\nabla^2 c^i(\xi_i) - \nabla^2 c^i(x)\| + \tfrac{1}{2} \|\lambda - \bar{\lambda}\| \, \|d\|^2 \sum_{i=1}^{m} \|\nabla^2 c^i(x)\| .$$

So, by the Standard Assumptions, since $\nabla^2 c^i$ and $g$ are uniformly continuous on $D$, it is clear that for small enough $\bar{\Delta}$, if $\|d_k\| \le \bar{\Delta}$ then

$$|f + g^T d + \tfrac{1}{2} d^T B d - f(x+d+w)| \le \beta \|d\|^2 ,$$

for any constant $\beta > 0$, and the desired result follows. □

If at some iterate $x_k$, the approximation to the Hessian of the Lagrangian, $Z_k^T B_k Z_k$, is not positive semi-definite, then Condition #2 implies that the predicted reduction will be greater than some fixed constant multiplied by the step length squared. Using this with Lemma 3, we can easily extend the proof of Theorem 1 to show that if the null-space components of the steps satisfy Conditions #1 and #2, then each inner iteration of the algorithm will terminate with an acceptable step after a finite number of iterations, unless the iterate satisfies the second order necessary conditions for a solution to (1.1). Note that if $c_k = 0$, then $v_k = 0$, so the second order correction will be tried if the trust region step is not accepted.

Now we show that the sequence of iterates can not converge to a point that does not satisfy the second order necessary conditions for a solution to (1.1). This result requires the second order correction to the trust region step, which ensures that the actual reduction in the merit function by the step from $x_k$ to $x_{k+1}$ will be close to as large as the predicted reduction indicated by a direction of negative curvature of the Hessian of the Lagrangian.

**Theorem 3**

Suppose that the Standard Assumptions hold, that $x_k \to x_*$, that

$$B_k = \nabla^2 f(x_k) + \sum_{i=1}^{m} \lambda^i(x_k) \nabla^2 c^i(x_k) ,$$

and that $\{\phi(x_k)\}$ is bounded below on $S$. Then if the null-space components $u$ in stage (7) of the algorithm satisfy Condition #1 and Condition #2, there is a $\lambda_*$ such that $g(x_*) + A(x_*)\lambda_* = 0$, $c(x_*) = 0$, and $Z(x_*)^T(\nabla^2 f(x_*) + \sum_{i=1}^{m} \lambda_*^i \nabla^2 c^i(x_*))Z(x_*)$ is positive semi-definite.

**Proof:**

By Theorem 2 and the Standard Assumptions, there is a $\lambda_*$ such that $g(x_*) + A(x_*)\lambda_* = 0$ and $\lambda(x_*) = \lambda_*$. Suppose to the contrary that the smallest eigenvalue of

$$Z(x_*)^T(\nabla^2 f(x_*) + \sum_{i=1}^{m} \lambda_*^i \nabla^2 c^i(x_*))Z(x_*)$$

is negative. Then there is some $r > 0$ and some $\omega > 0$ such that for any $x \in T$, where $T = \{x \in R^n : \|x - x_*\| < r\}$, we have $T$ contained in $S$ and

$$\nu_1(Z(x)^T(\nabla^2 f(x) + \sum_{i=1}^{m} \lambda^i(x) \nabla^2 c^i(x))Z(x)) < -\omega ,$$

by the continuity of $A$, $\lambda$, $\nabla^2 f$, and $\nabla^2 c^i$.

Now, consider an iterate $x \in T$. For the first case, suppose that $\|v\| \leq \zeta\Delta$. Then $\alpha = 1$ and $\tilde{\Delta} \geq (1-\zeta^2)^{1/2}\Delta$, so by Condition #2 and Lemma 1 there is a constant $\sigma_1$ such that $pred(d) \geq \sigma_1\Delta^2$ for any $x \in T$ and $\Delta > 0$. Now, if the trust region step $d$ is accepted, then $ared(d) \geq \eta\sigma_1\Delta^2$. Otherwise, the second order correction step $d+w$ will be tried, since $\|v\| \leq \zeta\Delta$. Thus, since by Lemma 3, for any $\epsilon > 0$, for all small enough $d$,

$$|ared(d+w) - pred(d)| \leq \epsilon\Delta^2 ,$$

and $pred(d) \geq \sigma_1\Delta^2$, the second order correction step $d+w$ will be accepted and yield $ared(d+w) \geq \eta\sigma_1\Delta^2$ for all small enough $\Delta > 0$.

The second case is that $\|v\| > \zeta\Delta$. In this case, by Lemma 1,

$$pred \geq \rho\alpha \|c\|_1 \geq \rho\min\{1, \theta\frac{\Delta}{\|v\|}\} \|c\|_1$$

$$\geq \rho \min\{1, \theta\frac{\Delta}{\|v\|}\}\frac{1}{\gamma_1}\|v\| \geq \rho \min\{\frac{\zeta}{\gamma_1}\Delta, \frac{\theta}{\gamma_1}\Delta\} .$$

Hence, by Lemma 2, the trust region step $d$ will be accepted and yield $pred \geq \sigma_2\Delta \geq \sigma_2\Delta^2$ for some $\sigma_2$ and for all small enough $\Delta > 0$.

Thus, for any $x \in T$, and any small enough $\Delta > 0$, either the trust region step will be accepted and yield

$$\frac{1}{\eta}ared(d) \geq pred(d) \geq \sigma_2\Delta^2 ,$$

or the second order correction step will be accepted and yield

$$\frac{1}{\eta}ared(d) \geq pred(d) \geq \sigma_1\Delta^2 .$$

Since $\{x_k\}$ converges to $x_*$, this clearly contradicts the assumption that $\phi$ is bounded below on $S$. □

The above theorem can be strengthened to to remove the assumption that $x_k \rightarrow x_*$ if the algorithm is changed slightly. In particular, suppose the strategy for choosing $\Delta$ in stage (2) of the algorithm is modified so that $\Delta$ is increased by at least a constant factor whenever *ared* and *pred* agree to a specified degree on the previous step. Then an argument like that in Theorem 3.1 of Shultz, Schnabel, and Byrd [1985] shows that at any cluster point of the algorithm, the reduced Hessian is positive semidefinite.

We now give a local rate of convergence result for the algorithm. When a good approximation to the Hessian of the Lagrangian is used, the second order correction allows us to prove that eventually the trust region constraint will become inactive in the neighborhood of a point that satisfies second order sufficient conditions for a solution to (1.1). Hence, if the sequence of iterates comes close enough to such a point, then the sequence will converge quadratically. In this paper the terms "quadratic convergence" or "superlinear convergence" refer to Q-order of convergence.

### Theorem 4

Let the Standard Assumptions hold. Let

$$B(x) = \nabla^2 f(x) + \sum_{i=1}^{m} \lambda^i(x) \nabla^2 c^i(x) ,$$

and $B_k = B(x_k)$. Suppose that $x_*$ is a point such that there exists a $\lambda_* \epsilon R^m$ with $g(x_*) + A(x_*)\lambda_* = 0$, $c(x_*) = 0$, and $Z_*^T B_* Z_*$ positive definite. Assume that in some neighborhood of $x_*$, $\nabla^2 f$ and each $\nabla^2 c^i$, for $i = 1,\ldots,m$ are Lipschitz continuous. Suppose further that Conditions #1 and #3 hold. Then there is a neighborhood about $x_*$ such that if any iterate falls in that neighborhood then $\{x_k\} \to x_*$, and the convergence rate is quadratic.

### Proof:

Since $Z_*^T B_* Z_*$ is positive definite, there is a neighborhood of $x_*$ and an $\omega > 0$ such that for any $x$ in the neighborhood, $\nu_1(Z(x)^T B(x) Z(x)) > \omega$.

We first show that $\mu^i \geq |\lambda_*^i|$ for all $i$. Note that since $c(x_*) = 0$, the Standard Assumptions imply that there is some neighborhood about $x_*$ in which

$$|\tilde{\mu}_k^i - \lambda^i(x_k)| \leq \frac{\rho}{2}$$

for each $i$, for any $x_k$ in that neighborhood. Also, since $\lambda(x)$ is uniformly continuous in some neighborhood of $x_*$, clearly there is some neighborhood of $x_*$ such that for each $i$,

$$|\lambda^i(x_k) - \lambda_*^i| \leq \frac{\rho}{2}$$

for any $x_k$ in that neighborhood. Thus, there is a neighborhood about $x_*$ such that if any iterate $x_k$ falls in that neighborhood, then

$$|\tilde{\mu}_k^i - \lambda_*^i| < \rho$$

for each $i$, and hence clearly

$$\mu^i \geq \tilde{\mu}_k^i + \rho \geq \lambda_*^i$$

as desired. Now, since $\mu \geq |\lambda_*|$, by Corollary 3 in Coleman and Conn [1980], we have that for all $x \neq x_*$ in some neighborhood of $x_*$, $\phi(x) > \phi(x_*)$.

Next we show that there is a neighborhood about $x_*$ such that if any iterate lands in this neighborhood, then the entire sequence of iterates will converge to $x_*$. Consider any $\delta > 0$, small enough that

$$\nu_1(Z(x)^T B(x) Z(x)) > \omega$$

and $\phi(x_*) < \phi(x)$ for all $x \neq x_*$ in

$$D = \{x \epsilon R^n : \|x - x_*\| < \delta\},$$

and with $\delta$ small enough that inside $D$, $\nabla^2 f$ and each $\nabla^2 c^i$ are Lipschitz continuous, if any iterate falls in $D$, then $\mu \geq |\lambda_*|$, and $\nu_1(Z(x)^T B(x) Z(x)) > \omega$ in $D$. For any $\sigma > 0$, let

$$E_\sigma = \{x \epsilon D : \phi(x) \leq \phi(x_*) + \sigma\}.$$

Clearly each $E_\sigma$ is a closed and bounded set. Note that

$$\lim_{\sigma \to 0^+} \max \{ \|x - x_*\| : x \in E_\sigma \} = 0 ,$$

since $x_*$ is a strict local minimum of $\phi$ over $D$. So, we can choose $\sigma > 0$ small enough that for any $x_k \in E_\sigma$, $\|x_k - x_*\| < ½\delta$. Further, since for $x_k \in D$, $\nu_1(Z_k^T B_k Z_k) > \omega > 0$, and $Z(x_*)^T g(x_*) = 0$, and $c(x_*) = 0$, it follows by Condition #3 that we can pick $\sigma$ small enough that if $x_k \in E_\sigma$, then $\|x_{k+1} - x_k\| < ½\delta$. Note that we can satisfy these conditions for any small enough $\sigma > 0$. Now, suppose that $x_k \in E_\sigma$. Since $\|x_{k+1} - x_k\| < ½\delta$, and $\|x_k - x_*\| < ½\delta$, it follows that $\|x_{k+1} - x_*\| < \delta$, so $x_{k+1} \in D$. Thus, since the algorithm ensures that $\phi(x_{k+1}) < \phi(x_k)$, $x_{k+1} \in E_\sigma$. So, we have shown that all the iterates will remain in $E_\sigma$, once any iterate falls in $E_\sigma$. Hence, since $E_\sigma$ is closed and bounded, the sequence of iterates must have an accumulation point in $E_\sigma$. But, since $\phi\{x_k\}$ is a monotone decreasing sequence and $x_*$ is a strict local minimum of $\phi$ over $E_\sigma$, the only possible accumulation point is $x_*$. Thus, $\{x_k\}$ converges to $x_*$ if some iterate falls in $E_\sigma$.

Now we will show that there is a $\bar{\Delta} > 0$ and a neighborhood of $x_*$ such that for any $x_k$ in the neighborhood and any $0 < \Delta < \bar{\Delta}$, the trust region step or the trust region step plus the second order correction will pass the corresponding trust region acceptance test.

If $\|v\| \geq \zeta \|d\|$, then by Lemma 1,

$$\begin{aligned} pred(d) &\geq \rho\alpha \|c\|_1 \\ &\geq \rho \min\{1, \theta \frac{\Delta}{\|v\|}\} \|c\|_1 \\ &\geq \rho \min\{\frac{\|v\|}{\gamma_1}, \frac{\theta}{\gamma_1} \Delta\} \\ &\geq \rho \min\{\frac{\zeta}{\gamma_1} \|d\|, \frac{\theta}{\gamma_1} \|d\|\} . \end{aligned}$$

But, by Lemma 2,

$$|ared(d) - pred(d)| \leq \gamma_2 \|d\|^2$$

for some constant $\gamma_2$. Hence, clearly for all small enough $\Delta$, if $\|v\| \geq \zeta \|d\|$, then $\frac{ared(d)}{pred(d)} \geq \eta$, and the trust region step $d$ will be accepted.

Otherwise, if $\|v\| < \zeta \|d\|$, and the step $d$ is not accepted, then the step $d + w$ will be tried. Now, for any $x_k$ close enough to $x_*$, $Z_k^T B_k Z_k$ is positive definite, so by Condition #3, either

$$u = -(Z^T BZ)^{-1} Z^T \tilde{g} ,$$

or

$$\|u\| \leq \tilde{\Delta} < \|(Z^T BZ)^{-1} Z^T \tilde{g}\| .$$

In either case,

$$\|u\| \leq \|(Z^T BZ)^{-1} Z^T \tilde{g}\| \leq \|(Z^T BZ)^{-1}\| \ \|Z^T \tilde{g}\| \leq \frac{1}{\omega} \|Z^T \tilde{g}\| .$$

By Lemma 1,

$$pred(d) \geq \kappa_1 \|Z^T \tilde{g}\| \min\{\tilde{\Delta} , \frac{\|Z^T \tilde{g}\|}{\|Z^T BZ\|}\}$$

$$\geq \kappa_1 \omega \|u\| \min\{ \|u\| , \frac{\omega}{\beta} \|u\|\}$$

$$\geq \gamma_5 \|d\|^2$$

for some constant $\gamma_5$, since

$$\|u\|^2 = \|d\|^2 - \alpha^2 \|v\|^2 \geq (1 - \zeta^2) \|d\|^2 .$$

Thus, since by Lemma 3, for any $\epsilon > 0$, for all small enough $d$,

$$|ared(d + w) - pred(d)| \leq \epsilon \|d\|^2 ,$$

it follows that the step $d + w$ from an iterate close enough to $x_*$ will be accepted for all $\Delta > 0$ less than some fixed $\overline{\Delta}$.

Thus, we have shown that there is some neighborhood of $x_*$ such that if any iterate falls in that neighborhood, then the sequence of iterates will converge to $x_*$, and the trust radii are bounded away from 0. So, since clearly $\|x_{k+1} - x_k\| \to 0$, the trust region constraint will eventually become inactive.

Finally we show that the convergence rate is quadratic. Since the trust region constraint is inactive, the trust region step $d_k$ is just the $x$-component of the Newton step at the point $(x_k, \lambda(x_k))$ on the problem

$$\nabla L(x,\lambda) = 0$$

$$c(x) = 0 ,$$

where $L(x,\lambda)$ is the Lagrangian function

$$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda^i(x) c^i(x) .$$

Since the Standard Assumptions imply that the Jacobian of the above non-linear equations problem is non-singular, and since we have assumed Lipschitz continuity, it follows by a standard result that the Newton iteration is quadratically convergent in the space of $x$ and $\lambda$. Thus there is a constant $\gamma_3$ such that

$$\|x_k + d_k - x_*\| \le \gamma_3( \|x_k - x_*\|^2 + \|\lambda(x_k) - \lambda(x_*)\|^2)$$

$$\le \gamma_4 \|x_k - x_*\|^2$$

for some $\gamma_4$, since the derivative of $\lambda(x)$ is bounded in a neighborhood of $x_*$.

Now, if the second order correction $w_k$ is used, the fact that

$$\|w_k\| \le C \|x_k + d_k - x_*\| ,$$

for some constant $C$, implies that

$$\|x_k + d_k + w_k - x_*\| \le (1 + C) \|x_k + d_k - x_*\|$$

$$\le (1 + C)\gamma_4 \|x_k - x_*\|^2 .$$

Hence, the convergence rate is quadratic. □

Although this result was proved for the case where $B_k$ is the exact Hessian, one may make similar claims given weaker assumptions on $B_k$. For example, if $B_k$ were constructed so that

$$\frac{\|Z_k^T(B_k - \nabla^2 f(x_k) + \sum_{i=1}^{m} \lambda^i(x_k)\nabla^2 c^i(x_k))(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|} \to 0 \qquad (4.1)$$

then an argument essentially identical to the preceding proof shows that, if $x_k \to x_*$, the trust region would eventually be inactive and convergence would be superlinear. However, we feel that such a result would be interesting only in the context of some practical quasi-Newton method which satisfied (4.5) while using a trust region.

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