

ADDING GLOBAL FORBIDDING  
CONTEXT TO CONTEXT-FREE GRAMMARS

by

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ADDING GLOBAL FORBIDDING  
CONTEXT TO CONTEXT-FREE GRAMMARS

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ABSTRACT

A 1S grammar generalizes a context-free grammar in the following way: a production  $A \rightarrow \alpha$  can be applied to a string  $uAv$  (to rewrite the designated occurrence of  $A$ ) provided that all letters from  $u$  belong to a fixed alphabet  $X$  and all letters from  $v$  belong to a fixed alphabet  $Z$  (the alphabets  $X$  and  $Z$  are independent of the production). It is proved that a language is generated by a 1S grammar if and only if it is context-free: this solves an open problem from the theory of selective substitution grammars [KR2].

KEYWORDS: Selective substitution grammars, one sequential grammars, context-free languages.

CR CATEGORIES: F.4.2, F.4.3.

## INTRODUCTION

The concept of a 1S grammar arose in a comparative study ([KR2]) of sequential and parallel ways of rewriting. This study was carried out within the theory of selective substitution grammars which provides a uniform framework for the grammatically oriented formal language theory (see, e.g., [R], [RW], [EMR], [KR1], [GKR1], [GKR2] and [K]). Within this framework 1S grammars form a generalization of Sequential rewriting (i.e. one symbol per step). The generalization consists of adding one specification of allowed left and right context sets of letters: an occurrence of a non-terminal in a sentential form cannot be rewritten if to its left or to its right a symbol appears which is not in the specified left or right, respectively, context set (a forbidden symbol).

Note that when no symbols are forbidden as left or right context, one gets the classical context-free way of rewriting.

The application of the context conditions is global: one set of rewritable non-terminals is specified together with left and right context sets which are independent of the productions to be used.

In a more local application of context conditions one specifies pairs of left and right context sets for various sets of non-terminals: if a letter from a given non-terminal set is to be rewritten, then its left-right context pair is taken into consideration.

It is shown in [KR2] that even the slightest local use of such context conditions increases the generative power of context-free grammars. However, it is left open in [KR2] whether 1S grammars are more powerful than context-free grammars. In the present paper we prove that a language is generated by a 1S grammar if and only if it is context-free.

It is worth noticing that the role of forbidding context in rewriting systems was investigated in depth within formal language theory (see, e.g., [vdW], [L], [M], [P], [RvS] and [C]). Although the notion of a 1S grammar

came up within the theory of selective substitution grammars, it clearly forms a very natural and perhaps the most basic way of adding forbidding context to context-free grammars fitting very well into systematizing the above mentioned line of research. In view of this our main result seems to be rather surprising - until now all "non-trivial" ways of adding forbidding context to context-free grammars resulted in a (considerable) increase of generating power.

The paper is organized as follows.

After the introduction of some basic terminology in Section 1, in Section 2 a subclass of 1S grammars called special 1S grammars is introduced. It is shown that every 1S language is a homomorphic image of the language generated by a special 1S grammar. In Section 3 the successful derivations in special 1S grammars are considered and it is demonstrated that for those a "normal form" exists. Finally, by constructing (in Section 4) special 1S grammars in which the successful derivations in "normal form" cannot get "blocked", we are able to show that every language generated by a special 1S grammar is context-free. This implies that every 1S grammar generates a context-free language (Section 5).

## 1. BASIC DEFINITIONS

We assume the reader to be familiar with the basic concepts of formal language theory as presented, e.g., in [S].

As far as selective substitution grammars are concerned the paper is self-contained.

In order to fix our notation and terminology we recall some basic notions now.

Let  $\Sigma$  be an alphabet. For a word  $w \in \Sigma^*$ ,  $|w|$  denotes its length and  $\text{alph}(w)$  denotes the set of symbols occurring in  $w$ ;  $\Delta$  denotes the empty word.

For a language  $L \subseteq \Sigma^*$ ,  $\text{alph}(L) = \bigcup_{w \in L} \text{alph}(w)$ .

For a binary relation  $P \subseteq \Sigma \times \Sigma^*$ , we write  $\text{rhs}(P) = \{w \in \Sigma^* : (A, w) \in P, \text{ for some } A \in \Sigma\}$ .

Two language generating devices are said to be equivalent, if the languages they generate are equal.

We often identify notationally a singleton set with its element.

Let  $\Sigma$  and  $\Gamma$  be two (finite) alphabets. A mapping  $h$  from  $\Sigma^*$  into the set of non-empty subsets of  $\Gamma^*$  is called a finite substitution from  $\Sigma$  into  $\Gamma$ ,

if, for all  $a \in \Sigma$ ,  $h(a)$  is finite and, for all  $a \in \Sigma$  and  $w \in \Sigma^*$ ,

$h(aw) = h(a)h(w)$ .  $h$  is a homomorphism from  $\Sigma$  into  $\Gamma$  if, additionally, for

all  $a \in \Sigma$ ,  $h(a)$  is a singleton.

By  $\text{FSUB}(\Sigma, \Gamma)$  we denote the set of all finite substitutions from  $\Sigma$  into  $\Gamma$  and

by  $\text{HOM}(\Sigma, \Gamma)$  the set of all homomorphisms from  $\Sigma$  into  $\Gamma$  is denoted.

For  $L \subseteq \Sigma^*$  and  $h \in \text{FSUB}(\Sigma, \Gamma)$ ,  $h(L)$  denotes the set  $\{h(w) : w \in L\}$ .

A context-free (abbreviated CF) grammar is specified in the form  $(\Sigma, P, S, \Delta)$ , where  $\Sigma$  is its total alphabet,  $\Delta \subseteq \Sigma$  is its terminal alphabet,  $S \in \Sigma \setminus \Delta$  is its axiom and  $P \subseteq (\Sigma \setminus \Delta) \times \Sigma^*$  is its (finite) set of productions.

For a context-free grammar  $G$ , its direct derivation relation and its derivation relation are as usual denoted by  $\Rightarrow$  and  $\xRightarrow{G}$ , respectively, and

if no confusion arises by  $\Rightarrow$  and  $\overset{*}{\Rightarrow}$ , respectively.

In the general theory of rewriting systems it is often essential to provide productions also for terminal symbols. If one allows to rewrite also terminal symbols in a context-free grammar then one gets an EOS system (see, e.g., [KR1]).

It is easily seen that  $L(\text{CF}) = L(\text{EOS})$ , where  $L(\text{CF})$  and  $L(\text{EOS})$  denote the families of languages generated by CF grammars and EOS systems, respectively.

In the sequel we will assume that an infinite alphabet of symbols is available. All symbols to be used are elements of the infinite alphabet  $A \cup \bar{A}$ , where  $\bar{A} = \{\bar{a} : a \in A\}$ , and  $A$  and  $\bar{A}$  are disjoint. The bars appearing above symbols have a special meaning: they indicate that the original symbol is activated.  $A$  consists of non-activated symbols only and hence  $\bar{A}$  consists of activated symbols only. For an arbitrary alphabet  $\Sigma \subseteq A$ ,  $\bar{\Sigma}$  denotes  $\{\bar{a} : a \in \Sigma\}$ .

Definition 1.1. (1). A CF based 1S grammar is a construct  $(\Sigma, P, S, \Delta, X \overset{*}{\bar{Y}} Z \overset{*}{})$ , where  $(\Sigma, P, S, \Delta)$  is a context-free grammar,  $X, Z \subseteq \Sigma$  and  $Y \subseteq \Sigma \setminus \Delta$ .

(2). An EOS based 1S grammar is a construct  $(\Sigma, P, S, \Delta, X \overset{*}{\bar{Y}} Z \overset{*}{})$ , where  $(\Sigma, P, S, \Delta)$  is an EOS system and  $X, Y, Z \subseteq \Sigma$ . □

For a Q based 1S grammar  $H = (\Sigma, P, S, \Delta, X \overset{*}{\bar{Y}} Z \overset{*}{})$ , where  $Q \in \{\text{CF}, \text{EOS}\}$ , we refer to  $(\Sigma, P, S, \Delta)$  as the base of  $H$ , denoted by base( $H$ ).

Definition 1.2. Let  $Q \in \{\text{CF}, \text{EOS}\}$ . Let  $H = (\Sigma, P, S, \Delta, X \overset{*}{\bar{Y}} Z \overset{*}{})$  be a Q based 1S grammar.

For  $u, v \in \Sigma^*$ ,  $u$  directly derives  $v$  (in  $H$ ), denoted by  $u \underset{H}{\Rightarrow} v$  ( $u \Rightarrow v$ , if no confusion is possible) if there exist

(1)  $u_1 \in X^*$ ,  $A \in Y$  and  $u_2 \in Z^*$ , and



(2)  $(A,w) \in P$ , such that  $u = u_1Au_2$  and  $v = u_1wu_2$ .

By  $\stackrel{*}{\Rightarrow}_H$  ( $\stackrel{*}{\Rightarrow}$ ) we denote the transitive and reflexive closure of  $\Rightarrow$  ( $\Rightarrow$ , respectively).

The language of H, denoted by  $L(H)$ , is defined by

$$L(H) = \{w \in \Delta^* : S \stackrel{*}{\Rightarrow}_H w\}.$$

□

In [KR2] only EOS based 1S grammars are considered. In the present paper, however our constructions involve CF based 1S grammars only. This is justified by the following straightforward lemma (see also [KR1]).

Lemma 1.1. (1). For every CF based 1S grammar there exists (effectively) an equivalent EOS based 1S grammar.

(2). For every EOS based 1S grammar there exists (effectively) an equivalent CF based 1S grammar.

□

In the remainder of this paper we will consider CF based 1S grammars only and we will refer to them as 1S grammars.

The family of languages generated by 1S grammars is denoted by  $L(1S)$ .

Next we introduce some notions and notations concerning 1S grammars, that will be extensively used in the sequel.

Let  $H = (\Sigma, P, S, \Delta, X \overset{*}{\bar{Y}} \overset{*}{Z})$  be a 1S grammar.

Then  $L_H = \Sigma \setminus Z$ ,  $R_H = \Sigma \setminus X$  and  $B_H = L_H \cap R_H$  are

the set of left-blocking symbols (of H),

the set of right-blocking symbols (of H), and

the set of blocking symbols (of H), respectively.

If a symbol is neither left-blocking, nor right-blocking (hence it is in  $X \cap Z$ ) we will refer to it as neutral (in H).

Example 1.1. Let  $H = (\Sigma, P, S, \Delta, X \overset{*}{\bar{Y}} \overset{*}{Z})$  be defined in the following way.

$$\Sigma = \{S, A, B, C, a, b, c\}.$$

$P = \{(S,ABC), (A,AA), (A,a), (A,c), (B,b), (C,CA), (C,BCC), (C,a), (C,c)\},$

$\Delta = \{a,b,c\}, X = \{a,b,A\}, Y = \{S,A,B,C\}$  and  $Z = \{c,b,C\}.$

Then  $L_H = \{S,A,B,a\}, R_H = \{S,B,C,c\}$  and  $B_H = \{S,B\}.$

Then  $S \Rightarrow ABC$  and in  $ABC$ ,  $B$  is the only symbol that can be rewritten. Since  $B$  is blocking it prohibits the rewriting of  $A$  and  $C$ .

$ABC \Rightarrow AbC \Rightarrow Abc \Rightarrow AAbc.$

In  $AAbc$  only the right-most  $A$  can be rewritten. Since  $A$  is left-blocking it prohibits the rewriting of symbols occurring to its left.

Analogously in  $ab^2CC$  (derived via  $AbC \Rightarrow abC \Rightarrow abBCC \Rightarrow ab^2CC$ ) only the left-most occurrence of  $C$  can be rewritten, because  $C$  is right-blocking.

In  $AbCA$  (derived from  $AbC$ ) no symbol can be rewritten because the right-most  $A$  "blocks"  $AbC$  and  $C$  "blocks" this  $A$ . □

Let  $w = a_1 \dots a_k$ , for some  $a_i \in \Sigma$  and  $1 \leq i \leq k$ , where  $k \geq 0$ .

Then  $Locc_H(w) = \max(\{i : 1 \leq i \leq k \text{ and } a_i \in (\Sigma \setminus \Delta) \cap L_H\} \cup \{0\})$

and  $Rocc_H(w) = \min(\{i : 1 \leq i \leq k \text{ and } a_i \in (\Sigma \setminus \Delta) \cap R_H\} \cup \{k+1\}).$

Hence  $Locc_H(w)$  denotes the right-most occurrence of a left-blocking non-terminal symbol in  $w$  if any, otherwise  $Locc_H(w) = 0$ , and  $Rocc_H(w)$  denotes the left-most occurrence of a right-blocking non-terminal symbol in  $w$  if any, otherwise  $Rocc_H(w) = |w| + 1$ .

Note that if  $Rocc_H(w) < Locc_H(w)$ , then  $w$  cannot be rewritten in  $H$ .

Whenever possible we will omit the subscript  $H$  from  $Locc_H$  and  $Rocc_H$ .

Example 1.1. (Continued.)

$Locc_H(ABC) = 2 = Rocc_H(ABC),$

$Locc_H(AbC) = 1$  and  $Rocc_H(AbC) = 3.$

$Locc_H(AAbc) = 2$  and  $Rocc_H(AAbc) = 5.$

$Locc_H(ab^2CC) = 0$  and  $Rocc_H(ab^2CC) = 4.$

$Locc_H(AbCA) = 4$  and  $Rocc_H(AbCA) = 3$  and  $AbCA$  is "blocked". □

Let  $w \in \Sigma^*$ . Then  $\text{contr}_H(w) = \{u : u \in \Delta^* \text{ and } w \xRightarrow[H]{*} u\}$ .

Notice that  $\text{contr}_H(w) \supseteq \{u : u \in \Delta^* \text{ and } w \xRightarrow[H]{*} u\}$ .

In particular  $\text{contr}_H(S) = L(\text{base}(H)) \supseteq L(H)$ .

A derivation  $D$  (in  $H$ ) is a sequence  $D = (x_0, x_1, \dots, x_n)$ , where  $n \geq 0$ ,  $x_0 = S$  and  $x_i \in \Sigma^*$  for  $0 \leq i \leq n$ , such that  $x_i \xRightarrow[H]{*} x_{i+1}$ , for  $0 \leq i \leq n-1$ . For  $0 \leq i \leq n-1$  the pair  $(x_i, x_{i+1})$  is called a (the  $(i+1)$ -th) derivation step (in  $D$ ).

For  $0 \leq i \leq n-1$ ,  $\text{Prod}(D, i)$  denotes the production used in the  $(i+1)$ -th derivation step in  $D$  and  $\text{Rew}(D, i)$  denotes the position of the symbol in  $x_i$  that is rewritten in that step. Thus if, for some  $0 \leq i \leq n-1$ ,  $\text{Prod}(D, i) = (A, \alpha)$  for some  $(A, \alpha) \in P$  and  $\text{Res}(D, i) = k$ , for some  $k \geq 1$ , then  $x_i = uAv$  and  $x_{i+1} = u\alpha v$ , where  $u, v \in \Sigma^*$  and  $|uA| = k$ .

$D$  is successful, if  $x_n \in \Delta^*$ .

For  $0 \leq i \leq n$ , the words  $x_i$  are said to appear in  $D$ .

Whenever a word appears in a derivation (in  $H$ ) it is called a sentential form (of  $H$ )

A non-terminal sentential form  $x$  of  $H$  is successful if there exists a  $w \in \Delta^*$  such that  $x \xRightarrow[H]{*} w$ .

Observe that if  $x$  is a successful non-terminal sentential form of  $H$ , then  $x \in X^*YZ^*$  and hence  $\text{Locc}_H(x) \subseteq \text{Rocc}_H(x)$ .

If moreover  $x = uAv$ , for some  $u, v \in \Sigma^*$  and  $A \in \Sigma \setminus \Delta$ , then  $(\text{alph}(u) \cap \Delta) \subseteq X$  and  $(\text{alph}(v) \cap \Delta) \subseteq Z$ . Consequently, for all  $w \in L(H)$ ,  $w \in \Delta^* \cap (X^* \text{rhs}(P) Z^*)$ .

Example 1.1. (Continued).

Let  $D = (S, ABC, AbC, Abc, AAbc, Acbc, acbc)$ . Then  $D$  is successful,  $(Abc, AAbc)$  is the fourth derivation step in  $D$ ,  $\text{Rew}(D, 3) = 1$  and  $\text{Prod}(D, 3) = (A, AA)$ .

The sentential form  $AAbc$  is successful, but  $Aabc$  and  $AbCA$  are not successful sentential forms. □

## 2. SPECIAL 1S GRAMMARS

In this section we show that every language in  $L(1S)$  is a homomorphic image of a language generated by a 1S grammar that satisfies certain conditions. To define this kind of 1S grammars we (re-)introduce the following notions.

First we consider context-free grammars (they are the underlying bases of 1S grammars).

Definition 2.1. Let  $G = (\Sigma, P, S, \Delta)$  be a context-free grammar and let  $A \in \Sigma$ .

(1).  $A$  is reachable (in  $G$ ), if there exist words  $\alpha, \beta \in \Sigma^*$  such that  $S \stackrel{*}{\Rightarrow} \alpha A \beta$ .

(2).  $A$  is useful (in  $G$ ), if either  $A = S$  or there exists a word  $\alpha \in \Delta^*$  such that  $A \stackrel{*}{\Rightarrow} \alpha$ .

(3).  $G$  is reduced if all elements of  $\Sigma$  are reachable and useful.  $\square$

The reader should note that this definition differs slightly from the usual definition of a reduced context-free grammar in that we consider the axiom always as useful, even if the generated language is empty.

Clearly in a 1S grammar all symbols appearing in a successful sentential form are reachable and useful.

Definition 2.2. Let  $H = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$  be a 1S grammar and let  $(A, \alpha) \in P$ .  $(A, \alpha)$  is safe (in  $H$ ) if the following holds.

(1). If  $\alpha \notin \Delta^*$ , then, for all  $\alpha_1, \alpha_2 \in \Sigma^*$  such that  $\alpha = \alpha_1 \alpha_2$  either  $\text{alph}(\alpha_1) \cap R_H = \emptyset$  or  $\text{alph}(\alpha_2) \cap L_H = \emptyset$ .

(2). If  $\alpha = \alpha_1 a \alpha_2$ , for some  $a \in \Delta$  and  $\alpha_1, \alpha_2 \in \Sigma^*$ , then  $a \in L_H$  implies that  $\alpha_1 \in \Delta^*$ , and  $a \in R_H$  implies that  $\alpha_2 \in \Delta^*$ .  $\square$

Clearly, in a 1S grammar a production that is not safe is never applied in a successful derivation.

Definition 2.3. A 1S grammar  $H = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$  is strongly reduced if the following holds.

- (1). base(H) is reduced.
- (2).  $Y = \Sigma \setminus \Delta$ .
- (3). All productions in P are safe. □

The following definition describes the construction of a strongly reduced 1S grammar from a given 1S grammar.

Definition 2.4. Let  $H = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$  be a 1S grammar. The strongly reduced version of H is the 1S grammar  $(\Sigma_1, P_1, S_1, \Delta_1, X_1^* \bar{Y}_1 Z_1^*)$  is constructed in the following way.

- (1).  $P' = P \setminus \{(A, w) : (A, w) \in P \text{ and either } A \notin Y \text{ or } (A, w) \text{ is not safe in } H\}$ .
- (2). Let G be the CF grammar  $(\Sigma, P', S, \Delta)$ .

Then  $\Sigma_1 = \{A \in \Sigma : A \text{ is useful and reachable in } G\}$  and  $\Delta_1 = \Delta \cap \Sigma_1$ .

- (3).  $P_1 = \{(A, w) : (A, w) \in P' \text{ and } A, w \in \Sigma_1^*\}$ ,  $S_1 = S$ ,  $X_1 = X \cap \Sigma_1$ ,  
 $Y_1 = (Y \cap \Sigma_1) \cup \{S_1\}$  and  $Z_1 = Z \cap \Sigma_1$ . □

Clearly the procedure given by Definition 2.4 is effective. Note that the resulting 1S grammar may be quite "degenerated", e.g. of the form  $(\{S\}, \emptyset, S, \emptyset, \bar{S})$ .

The following lemma is an immediate consequence of the preceding definitions and so we state it without a proof.

Lemma 2.1. If H is a 1S grammar and H' is its strongly reduced version, then  $L(H) = L(H')$  and H' is strongly reduced. □

Now we are ready to define one of the main notions of this paper and to prove an important intermediate result.

Definition 2.5. A 1S grammar  $G = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$  is special if the following conditions are satisfied.

$$(C1) \quad L(G) \subseteq (\Delta \setminus B_G)^* B_G (\Delta \setminus B_G)^*$$

(i.e., every word in  $L(G)$  contains exactly one occurrence of a blocking symbol).

$$(C2) \quad \text{For all } \alpha \in \text{rhs}(P), \text{ if } \underline{\text{alph}}(\alpha) \cap (B_G \cap \Delta) \neq \emptyset, \text{ then } \alpha \in (B_G \cap \Delta)$$

(i.e., blocking terminal symbols are never introduced together with another symbol).

$$(C3) \quad X \cap Z \subseteq \Delta$$

(i.e., there are no neutral non-terminal symbols in  $G$ ).

$$(C4) \quad S \notin \underline{\text{alph}}(\text{rhs}(P)) \text{ and, for all } A \in \Sigma \setminus (\Delta \cup \{S\}), \text{ for all } v, w \in \text{contr}_G(A),$$

$$\underline{\text{alph}}(v) = \underline{\text{alph}}(w)$$

(i.e.,  $S$  cannot be introduced during a derivation and all terminal words contributed by a non-terminal differing from  $S$  consist of the same symbols).

$$(C5) \quad \text{For all } \alpha_1 A \alpha_2 \in \text{rhs}(P), \text{ such that } \alpha_1, \alpha_2 \in \Sigma^* \text{ and } A \in \Sigma \setminus \Delta, \text{ if}$$

$$\underline{\text{alph}}(\text{contr}_G(A)) \cap B_G \neq \emptyset, \text{ then } \underline{\text{contr}}_G(\alpha_1) \subseteq X^* \text{ and } \underline{\text{contr}}_G(\alpha_2) \subseteq Z^*$$

(i.e., if a non-terminal symbol  $A$  can derive a blocking terminal symbol, then  $A$  is introduced with left and right context that contribute only terminal words which cannot block the rewriting of  $A$  or the rewriting of words derivable from  $A$ ).

$$(C6) \quad G \text{ is strongly reduced}$$

(i.e.,  $G$  has no a priori useless productions or symbols). □

Theorem 2.1. For every 1S grammar  $H$  there exist a special 1S grammar  $G$  and a homomorphism  $h$ , such that  $h(L(G)) = L(H)$ .

Proof. We will prove the theorem in five steps.

In the first step a 1S grammar  $G_1$  is constructed from  $H$  and the homomorphism  $h$  is defined for which  $h(L(G_1)) = L(H)$ .  $G_1$  will satisfy (C1) and (C2) from Definition 2.5.

In the next steps we will construct the 1S grammars  $G_2, G_3, G_4$  and  $G$  from  $G_1, G_2, G_3$  and  $G_4$ , respectively, in such a way that gradually all conditions from Definition 2.5 will be satisfied and the languages  $L(G_i), i = 1, \dots, 4$  and  $L(G)$  will be the same.

Then  $G$  will be special and  $h(L(G)) = L(H)$ .

Let  $H = (\Sigma, P, S, \Delta, X \bar{Y} Z^{\#})$  be a 1S grammar.

(I) Let  $G_1 = (\Sigma_1, P_1, S_1, \Delta_1, X_1 \bar{Y} Z_1^{\#})$  be defined in the following way.

$\Sigma_1 = \Sigma \cup \Gamma \cup \Pi$ , where  $\Gamma = \{\hat{A} : A \in \Sigma \setminus \Delta\}$  and  $\Pi = \{\langle w \rangle : w \in \Delta^{\#} \text{ and}$

$(A, w) \in P, \text{ for some } A \in \Sigma \setminus \Delta\}$  are new mutually disjoint alphabets.

$P_1 = \{(A, w) : (A, w) \in P \text{ and if } |w| \geq 2, \text{ then } \underline{\text{alph}}(w) \cap (\Delta \cap B_H) = \emptyset\} \cup$   
 $\cup \{(\hat{A}, w_1 \hat{B} w_2) : (A, w_1 B w_2) \in P \text{ and } B \in \Sigma \setminus \Delta\} \cup \{(\hat{A}, \langle w \rangle) : (A, w) \in P \text{ and } w \in \Delta^{\#}\}.$

$S_1 = \hat{S}.$

$\Delta_1 = \Delta \cup \Pi,$

$X_1 = X \cup \{\hat{A} : A \in X \cap (\Sigma \setminus \Delta)\},$

$Y_1 = Y \cup \{\hat{A} : A \in Y \cap (\Sigma \setminus \Delta)\}$  and

$Z_1 = Z \cup \{\hat{A} : A \in Z \cap (\Sigma \setminus \Delta)\}.$

Clearly,  $L(G_1) \subseteq \Delta^{\#} \Pi \Delta^{\#}$ . Moreover, since  $\Pi \subseteq \Sigma_1 \setminus (X_1 \cup Z_1)$ , all elements from  $\Pi$  are blocking in  $G_1$ . In every successful derivation blocking terminals can only be introduced in the last derivation step. Hence this is done using a production of the form  $(\hat{A}, \langle w \rangle)$ , where  $\langle w \rangle \in \Pi$ .

This implies that  $L(G_1) \subseteq (\Delta_1 \setminus B_{G_1})^{\#} B_{G_1} (\Delta_1 \setminus B_{G_1})^{\#}$  and so  $G_1$  satisfies (C1).

From the construction of  $P_1$  it follows that  $G_1$  satisfies (C2).

The homomorphism  $h \in \text{HOM}(\Delta_1, \Delta)$  is defined by  $h(a) = a$ , for all  $a \in \Delta$ ,

and  $h(\langle w \rangle) = w$ , for all  $w \in \Delta^{**}$ .

That  $h(L(G_1)) = L(H)$  follows from the following observations.

Let  $(S_1, x_1, \dots, x_n)$  be a successful derivation in  $G_1$ . Then  $(\psi(S_1), \psi(x_1), \dots, \psi(x_n))$  is a successful derivation in  $H$ , where  $\psi \in \text{HOM}(\Sigma_1, \Sigma)$  is defined by

$$\psi(\hat{A}) = \psi(A) = A, \text{ for all } A \in \Sigma_1 \setminus \Delta_1 \text{ and}$$

$$\psi(a) = h(a), \text{ for all } a \in \Delta_1.$$

Hence  $h(L(G_1)) \subseteq L(H)$ .

Let  $D = (x_0 = S, x_1, \dots, x_n)$  be a successful derivation in  $H$ . Then  $x_{n-1} = uAv$  and  $x_n = uvw$ , for some  $u, v, w \in \Delta^{**}$ ,  $A \in \Sigma \setminus \Delta$  and  $(A, w) \in P$ .

A derivation  $\hat{D} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_n)$  in  $G_1$  with  $\hat{x}_0 = \hat{S}$ ,  $\hat{x}_{n-1} = u\hat{A}v$  and  $\hat{x}_n = u\langle w \rangle v$ , can be defined in inductively by "capping all ancestors of  $\hat{A}$ ".

This can be illustrated as follows.

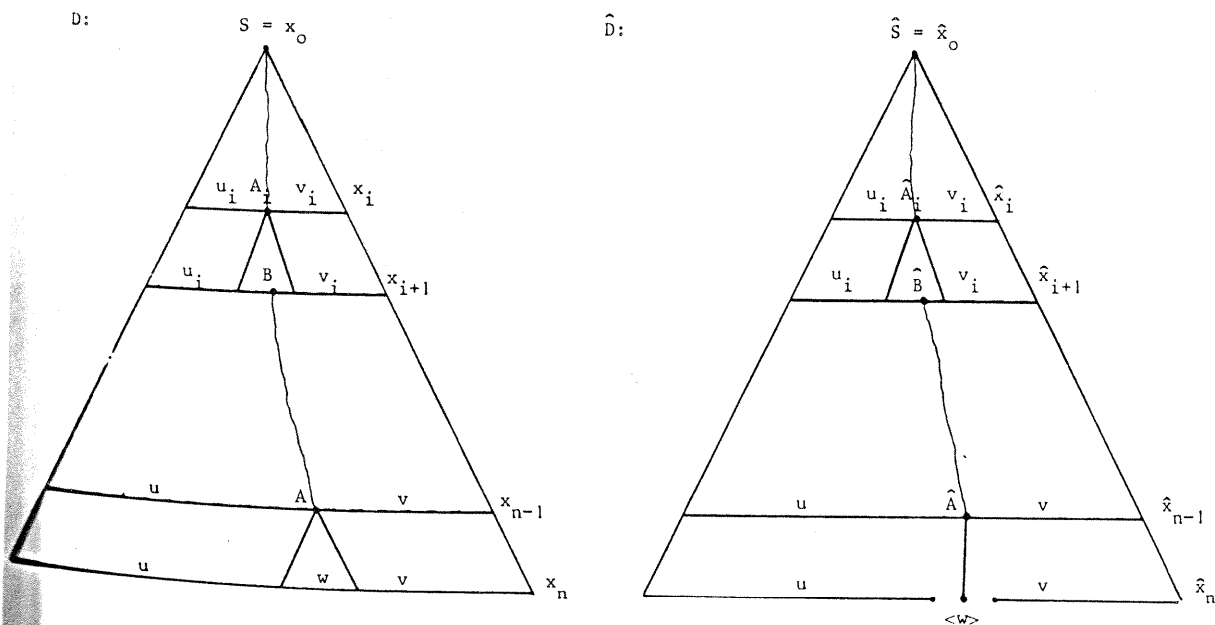


Figure 2.1.



Then  $\hat{D}$  is a successful derivation in  $G_1$  and  $h(\hat{x}_n) = x_n$ .

Hence  $L(H) \subseteq h(L(G_1))$ .

We conclude that  $h(L(G_1)) = L(H)$  and  $G_1$  satisfies (C1) and (C2).

(II) Let  $G_2 = (\Sigma_2, P_2, S_2, \Delta_2, X_2, Y_2, Z_2)$  be defined in the following way.

$\Sigma_2 = \Sigma_1 \setminus ((X_1 \cap Z_1) \setminus \Delta_1) \cup \Gamma_X \cup \Gamma_Z$ , where  $\Gamma_X = \{[A,1] : A \in (X_1 \cap Z_1) \setminus \Delta_1\}$  and

$\Gamma_Z = \{[A,2] : A \in (X_1 \cap Z_1) \setminus \Delta_1\}$  are two new mutually disjoint alphabets.

The homomorphism  $\varphi \in \text{HOM}(\Sigma_2, \Sigma_1)$  is defined by

$\varphi(a) = a$ , for all  $a \in \Sigma_1 \cap \Sigma_2$ , and

$\varphi([A,i]) = A$ , for all  $A \in (X_1 \cap Z_1) \setminus \Delta_1$  and  $i \in \{1,2\}$ .

$Q_2 = Q_1 \cup Q_2$ , where  $Q_1 = \{(A, \alpha) : A \in \Sigma_2 \text{ and } (\varphi(A), \varphi(\alpha)) \in P_1\}$  and

$Q_2 = \{([A,1], [A,2]) : A \in (X_1 \cap Z_1) \setminus \Delta_1\} \cup \{([A,2], [A,1]) : A \in (X_1 \cap Z_1) \setminus \Delta_1\}$ .

If  $S_1 \in \Sigma_2$ , then  $S_2 = S_1$  else  $S_2 = [S_1,1]$ .  $\Delta_2 = \Delta_1$ .

$X_2 = (X_1 \cap \Sigma_2) \cup \Gamma_X$ ,

$Y_2 = (Y_1 \cap \Sigma_2) \cup \{[A,i] : A \in Y_1 \cap X_1 \cap Z_1 \text{ and } i \in \{1,2\}\}$  and

$Z_2 = (Z_1 \cap \Sigma_2) \cup \Gamma_Z$ .

From the construction of  $G_2$  it follows that  $X_2 \cap Z_2 \subseteq \Delta_2$ . Hence  $G_2$  has only left-blocking and right-blocking non-terminal symbols and satisfies (C3) of Definition 2.5.

$G_2$  satisfies (C2), because  $G_1$  satisfies (C2), the blocking symbols of  $G_1$  and  $G_2$  are the same,  $Q_1$  "corresponds" to  $P_1$  and no productions violating (C2) have been included in  $Q_2$ .

If  $L(G_1) = L(G_2)$ , then  $G_2$  satisfies (C1) because the blocking terminals in  $G_1$  and  $G_2$  are the same.

That  $L(G_2) = L(G_1)$  can be seen as follows.

Let  $D = (x_0 = S_1, x_1, \dots, x_n)$  be a successful derivation in  $G_2$ . Clearly, for every  $i \in \{0, \dots, n-1\}$  such that  $\text{Prod}(D, i) \in Q_1$ ,  $\varphi(x_i)$  derives  $\varphi(x_{i+1})$  in  $G_1$ . For every  $i \in \{0, \dots, n-1\}$  such that  $\text{Prod}(D, i) \in Q_2$ ,  $\varphi(x_i) = \varphi(x_{i+1})$ . Let  $i_1, \dots, i_k$ ,  $k \geq 1$ , be - in ascending order - the

elements of  $\{i : i \in \{0, \dots, n-1\} \text{ and } \text{Prod}(D, i) \in Q_1\}$ . Then  $\varphi(x_{i_1}) = S_1$  and  $(\varphi(x_{i_1}), \dots, \varphi(x_{i_k}), x_n)$  is a derivation in  $G_1$ .

Hence  $L(G_2) \subseteq L(G_1)$ .

To show that  $L(G_1) \subseteq L(G_2)$  we proceed as follows.

Let  $D_1 = (x_0, x_1, \dots, x_n)$  be a successful derivation in  $G_1$ . We construct a successful derivation  $D_2 = (y_0, y_1, \dots, y_m)$  in  $G_2$  and a monotonic function  $\delta$  from  $\{0, \dots, n\}$  to  $\{0, \dots, m\}$  such that for all  $i \in \{0, \dots, n\}$ ,

$$\varphi(y_{\delta(i)}) = x_i \text{ and } \text{Locc}_{G_2}(y_{\delta(i)}) \leq \text{Rocc}_{G_2}(y_{\delta(i)}).$$

We will define  $\delta$  and  $D_2$  inductively.

We set  $\delta(0) = 0$  and  $y_0 = S_2$ . Hence  $\varphi(y_{\delta(0)}) = x_0$  and

$$\text{Locc}_{G_2}(y_{\delta(0)}) \leq \text{Rocc}_{G_2}(y_{\delta(0)}).$$

Assume that  $\delta$  and  $D_2$  have been defined upto  $k$  and  $y_{\delta(k)}$ , for some  $k \in \{0, \dots, n-1\}$ .

Let  $\text{Rew}(D_1, k) = p$ , for some  $1 \leq p \leq |x_k|$  and let  $\text{Prod}(D_1, k) = (A, \alpha)$ , for some  $(A, \alpha) \in P_1$ . Hence  $x_k = uAv$  and  $x_{k+1} = u\alpha v$ , for some  $u, v \in \Sigma_1^*$  such that  $|uA| = p$ . Then  $y_{\delta(k)} = u'A'v'$ , with  $\varphi(u') = u$ ,  $\varphi(A') = A$  and  $\varphi(v') = v$ .

Let  $(A', \alpha')$  be a production in  $P_2$  such that  $\varphi(\alpha') = \alpha$  and such that  $\text{Locc}_{G_2}(\alpha') \leq \text{Rocc}_{G_2}(\alpha')$ . Since  $D_1$  is a successful derivation in  $G_1$ ,  $(A, \alpha)$  is a safe production and hence a production  $(A', \alpha')$  as described exists.

Note that  $A' \in Y_2$ .

We distinguish the following cases.

(1).  $u' \in X_2^*$  and  $v' \in Z_2^*$ .

In this case  $y_{\delta(k)} = u'A'v' \xrightarrow{G_2} u'\alpha'v'$ . We set  $y_{\delta(k)+1} = u'\alpha'v'$  and  $\delta(k+1) = \delta(k) + 1$ . Then  $\varphi(y_{\delta(k+1)}) = \varphi(u'\alpha'v') = u\alpha v = x_{k+1}$  and

$$\text{Locc}_{G_2}(y_{\delta(k+1)}) \leq \text{Rocc}_{G_2}(y_{\delta(k+1)}).$$

$(y_0, \dots, y_{\delta(k)}, y_{\delta(k+1)})$  is a derivation in  $G_2$ .

(2).  $u' \notin X_2^*$  or  $v' \notin Z_2^*$ .

(2.1).  $u' \notin X_2^*$ .

Since  $\varphi(u') = u$  and  $u \in X_1^*$ ,  $u'$  contains at least one symbol from  $\Gamma_Z$ .

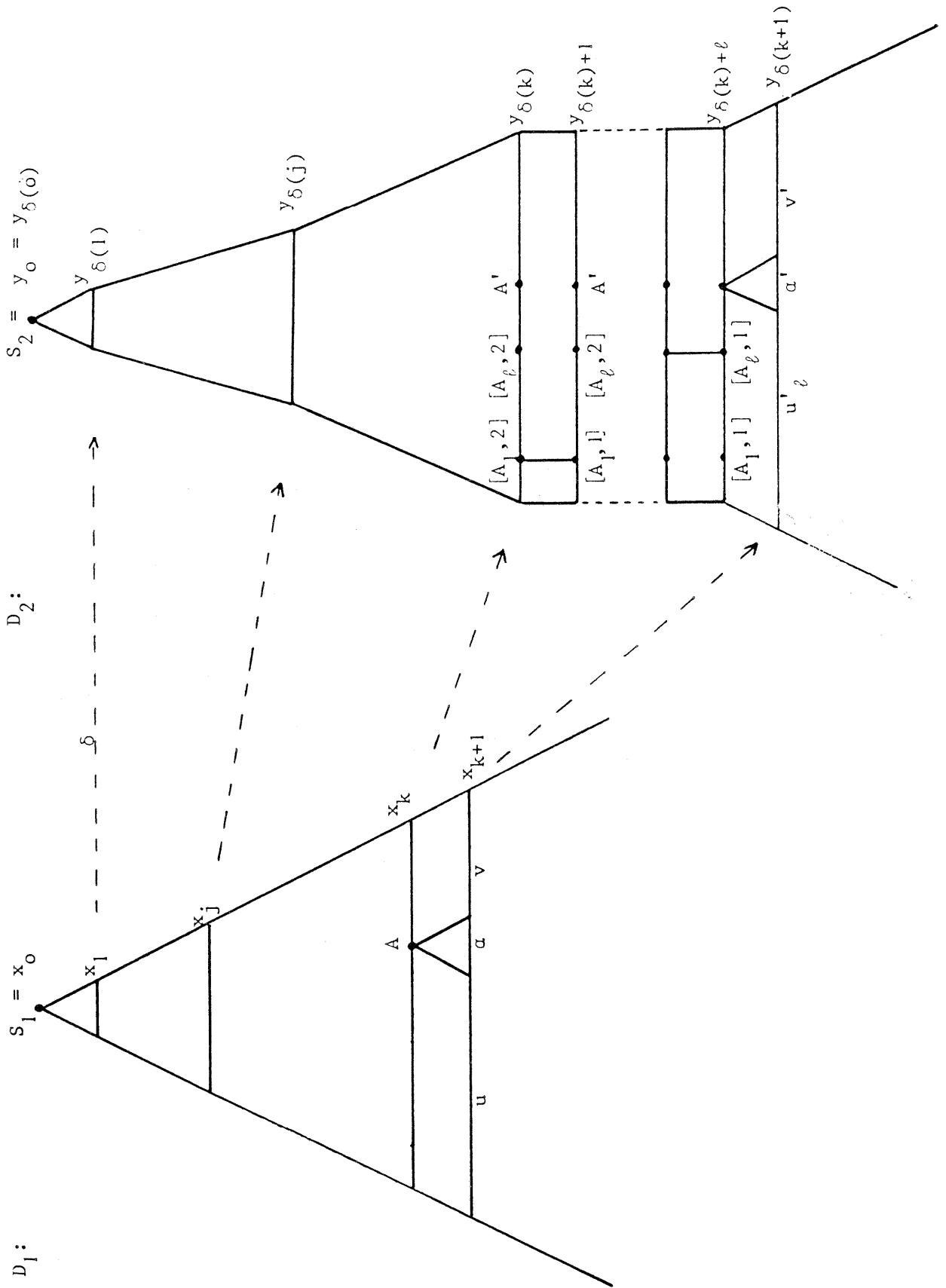


Figure 2.2

Since  $\text{Locc}_{G_2}(y_{\delta(k)}) \leq \text{Rocc}_{G_2}(y_{\delta(k)})$ , it follows that  $A' \in Z_2$  and, since  $\varphi(v') = v$  with  $v \in Z_1^*$ ,  $v' \in Z_2^*$ . Moreover,  $u = u_1[A_1,2]u_2[A_2,2]\dots u_\ell[A_\ell,2]u_{\ell+1}$ , for some  $u_1 \in X_1^*$ ,  $u_2, \dots, u_{\ell+1} \in (\Delta_1 \cap (X_1 \cap Z_1))^*$  and  $[A_i,2] \in \Gamma_Z$ , for  $1 \leq i \leq \ell$ .

Then we define  $u'_j$ ,  $1 \leq j \leq \ell$  as follows

$$u'_j = u_1[A_1,1]u_2 \dots u_j[A_j,1]u_{j+1}[A_{j+1},2]u_{j+2} \dots u_\ell[A_\ell,2]u_{\ell+1}.$$

Let  $y_{\delta(k)+j} = u'_j A' v'$ , for  $1 \leq j \leq \ell$ .

Then  $y_{\delta(k)+j-1} \stackrel{G_2}{=} y_{\delta(k)+j}$ , for  $1 \leq j \leq \ell$ . Moreover,  $y_{\delta(k)+\ell} = u'_\ell A' v'$  and

$y_{\delta(k)+\ell} \stackrel{G_2}{=} u'_\ell A' v'$ , where  $\varphi(u'_\ell) = u$ .

We set  $\delta(k+1) = \delta(k) + \ell + 1$  and  $y_{\delta(k+1)} = u'_\ell A' v'$ . Hence  $\varphi(y_{\delta(k+1)}) = x_{k+1}$

and  $\text{Locc}_{G_2}(y_{\delta(k+1)}) \leq \text{Rocc}_{G_2}(y_{\delta(k+1)}) \cdot (y_0, \dots, y_{\delta(k)}, y_{\delta(k)+1}, \dots, y_{\delta(k)+\ell}, y_{\delta(k+1)})$

is a derivation in  $G_2$ . This reasoning is illustrated in Figure 2.2.

(2.ii) The case that  $v' \notin Z_2^*$  is symmetric to (2.i) and can be treated analogously.

Now, let  $m = \delta(n)$ . Then  $\varphi(y_m) = x_n$  and  $(y_0, y_1, \dots, y_m)$  is a successful derivation in  $G_2$ .

Hence  $L(G_1) \subseteq L(G_2)$ .

This implies that  $L(G_2) = L(G_1)$  and that  $G_1$  satisfies (C1).

We conclude that  $G_2$  satisfies (C1), (C2) and (C3), and  $L(G_2) = L(G_1)$ .

(III). Let  $G_3 = (\Sigma_3, P_3, S_3, \Delta_3, X_3^*, Y_3^*, Z_3^*)$  be defined in the following way.

$\Sigma_3 = \{[A, T] : A \in \Sigma_2 \setminus \Delta_2 \text{ and } T \subseteq \Delta_2\} \cup \{S_3\} \cup \Delta_3$ , where  $\Delta_3 = \Delta_2$  and  $S_3$  is a new symbol.

The homomorphism  $\rho \in \text{HOM}(\Sigma_3 \setminus \{S_3\}, \Sigma_2)$  is defined by

$\rho(a) = a$ , for all  $a \in \Delta_3$ , and

$\rho([A, T]) = A$ , for all  $A \in \Sigma_2 \setminus \Delta_2$  and  $T \subseteq \Delta_2$ .

The finite substitution  $\tau \in \text{FSUB}(\Sigma_3 \setminus \{S_3\}, \Delta_2)$  is defined by

$\tau(a) = a$ , for all  $a \in \Delta_3$ , and

$\tau([A, T]) = T$ , for all  $A \in \Sigma_2 \setminus \Delta_2$  and  $T \subseteq \Delta_2$ .

$$P_3 = \{(S_3, [S_2, T]) : T \subseteq \Delta_2\} \cup \{(A, w) : A \in \Sigma_3 \setminus \Delta_3, w \in \Sigma_3^*, (\rho(A), \rho(w)) \in P_2 \text{ and } \tau(A) = \underline{\text{alph}}(\tau(w))\}.$$

$$X_3 = \rho^{-1}(X_2),$$

$$Y_3 = \rho^{-1}(Y_2) \cup \{S_3\} \text{ and}$$

$$Z_3 = \rho^{-1}(Z_2).$$

Note that, for  $[A, T]$  with  $A \in \Sigma_2 \setminus \Delta_2$  and  $T \subseteq \Delta_2$ ,  $T$  "promises" what terminal symbols will be contributed by  $[A, T]$ .

Firstly we prove that  $L(G_3) = L(G_2)$ .

Let  $(S_3, x_1, \dots, x_n)$  be a successful derivation in  $G_3$ . Then  $\rho(x_1) = S_2$  and  $\rho(x_n) = x_n$ . From the construction of  $G_3$  it follows immediately that  $(\rho(x_1), \dots, \rho(x_n))$  is a derivation in  $G_2$ .

Thus  $L(G_3) \subseteq L(G_2)$ .

To prove the converse inclusion we proceed as follows.

Let  $D = (x_0, x_1, \dots, x_n)$  be a successful derivation in  $G_2$ .

Then we set  $\hat{x}_n = x_n$ . Hence  $\rho(\hat{x}_n) = x_n$ .

Assume that for some  $k \in \{1, \dots, n\}$ ,  $\hat{x}_k, \dots, \hat{x}_n$  have been defined in such a way that  $\rho(\hat{x}_j) = x_j, k \leq j \leq n$ , and  $\hat{x}_j \xrightarrow{G_3} \hat{x}_{j+1}, k \leq j \leq n-1$ .

Let  $\text{Prod}(D, k-1) = (A, \alpha)$  for some  $(A, \alpha) \in P_2$ . Hence  $\hat{x}_k = \hat{u}\hat{\alpha}\hat{v}$ , for some  $\hat{u}, \hat{\alpha}, \hat{v} \in \Sigma_3^*$  such that  $\rho(\hat{\alpha}) = \alpha$  and  $\text{Rew}(D, k-1) = |\hat{u}| + 1$ . Let  $T = \underline{\text{alph}}(\tau(\hat{\alpha}))$ .

Then  $([A, T], \hat{\alpha}) \in P_3$ .

Let  $\hat{x}_{k-1} = \hat{u}[A, T]\hat{v}$ . Hence  $\rho(\hat{x}_{k-1}) = x_{k-1}$  and  $x_{k-1} \xrightarrow{G_2} x_k$ .

Then  $(\hat{S}_3, \hat{x}_0, \hat{x}_1, \dots, \hat{x}_n)$  is a derivation in  $G_3$ .

Thus  $L(G_3) \supseteq L(G_2)$

Together with the above we have  $L(G_3) = L(G_2)$ .

From the construction of  $P_3$  it follows that  $S_3 \notin \underline{\text{alph}}(\text{rhs}(P_3))$ .

Moreover, clearly for every  $B \in \Sigma_3$ , where  $B = [A, T]$ , for some  $A \in \Sigma_2 \setminus \Delta_2$  and  $T \subseteq \Delta_2$ ,  $\text{contr}_{G_3}(B) \subseteq \{w \in T^* : \underline{\text{alph}}(w) = T\}$ .

Hence  $G_3$  satisfies (C4).

That  $G_2$  also satisfies (C1), (C2) and (C3) can be shown using the following observations.

- (1).  $X_3 \cap \Delta_3 = X_2 \cap \Delta_2$ ,  $Z_3 \cap \Delta_3 = Z_2 \cap \Delta_2$ ,  $L(G_2) = L(G_3)$  and  $G_2$  satisfies (C1). Hence (C1) holds for  $G_3$ .
- (2). From the construction of  $P_3$  from  $P_2$  and arguments similar to the above, it follows that  $G_3$  satisfies (C2).
- (3).  $X_3 \cap Z_3 = \rho^{-1}(X_2) \cap \rho^{-1}(Z_2) = \rho^{-1}(X_2 \cap Z_2) \subseteq \Delta_2$  and hence (C3) holds for  $G_3$ .

We conclude that  $G_3$  satisfies (C1), (C2), (C3) and (C4), and  $L(G_3) = L(G_2)$ .

(IV) Let  $G_4 = (\Sigma_4, P_4, S_4, \Delta_4, X_4, Y_4, Z_4)$  be defined in the following way.  
 $\Sigma_4 = \Sigma_3$ ,  $S_4 = S_3$ ,  $\Delta_4 = \Delta_3$ ,  $X_4 = X_3$ ,  $Y_4 = Y_3$ ,  $Z_4 = Z_3$  and  
 $P_4 = P_3 \setminus \{(C, \alpha_1 A \alpha_2) : A \in \theta, \text{contr}_{G_3}(\alpha_1) \not\subseteq X_3 \text{ or } \text{contr}_{G_3}(\alpha_2) \not\subseteq Z_3\}$ ,  
 where  $\theta = \{A : A \in \Sigma_3 \setminus (\Delta_3 \cup \{S_3\}) \text{ and } \text{contr}_{G_3}(A) \subseteq \Delta_3 B_{G_3} \Delta_3\}$ .

Since  $G_4$  results from  $G_3$  by removing productions from  $P_3$  we have  $L(G_4) \subseteq L(G_3)$ .

We will prove the converse inclusion by showing that no production from  $P_3 \setminus P_4$  can be used in a successful derivation in  $G_3$ .

Let  $x$  and  $y$  be sentential forms of  $G_3$  such that  $x \Rightarrow y$  using a production from  $P_3 \setminus P_4$ . Hence  $x = uCv$  and  $y = u\alpha_1 A \alpha_2 v$ , for some  $u, v \in \Sigma_3^*$ ,  $C, A \in \theta$  and  $(C, \alpha_1 A \alpha_2) \in P_3 \setminus P_4$ .

Then every  $w \in \text{contr}_{G_3}(y)$  is of the form  $u_1 \alpha_1' \beta_1 b \beta_2 \alpha_2' v_1$ , where  $u_1 \in \text{contr}_{G_3}(u)$ ,  $\alpha_i' \in \text{contr}_{G_3}(\alpha_i)$ ,  $i = 1, 2$ ,  $\beta_1 b \beta_2 \in \text{contr}_{G_3}(A)$ , where  $b$  is a blocking terminal and  $v_1 \in \text{contr}_{G_3}(v)$ .

Observe that, since  $G_3$  satisfies (C4), for all  $A \in \Sigma_3 \setminus \Delta_3$  and  $T \in \Delta_3$ , whenever  $\alpha \notin T^*$ , for some  $\alpha \in \text{contr}_{G_3}(A)$ , then, for all  $\beta \in \text{contr}_{G_3}(A)$ ,  $\beta \notin T^*$ .

Hence  $\alpha_1' \notin X_3^*$  or  $\alpha_2' \notin Z_3^*$  and so  $w$  cannot be in  $L(G_3)$  which is a subset of  $(X_3 \cap \Delta_3)^* B_{G_3} (Z_3 \cap \Delta_3)^*$ .

Thus no derivation in  $G_3$  that uses a production from  $P_3 \setminus P_4$  is successful.

Hence  $L(G_3) \subseteq L(G_4)$ .

$G_3$  satisfies (C1) through (C4) and  $G_4$  results from  $G_3$  by removing the productions that violate (C5) - hence  $G_4$  satisfies (C1) through (C5).

We conclude that  $G_4$  satisfies (C1), (C2), (C3), (C4) and (C5), and  $L(G_4) = L(G_3)$ .

(V) Let  $G$  be the strongly reduced version of  $G_4$ . Then  $G$  and the homomorphism  $h$  (defined in (I)) satisfy the statement of the theorem because, by Lemma 2.1,  $L(G) = L(G_4)$  and hence  $L(G) = L(G_1)$ , and  $L(G_1) = h(L(G))$ .  $G$  is strongly reduced.

$G_4$  satisfies (C1) through (C5) and since  $G$  results from  $G_4$  by removing useless symbols and productions,  $G$  also satisfies (C1) through (C5).

We conclude that  $G$  is special and  $h(L(G)) = L(H)$ . □

### 3. CENTRAL DERIVATIONS IN SPECIAL 1S GRAMMARS

In this section we concentrate on (successful) derivations in special 1S grammars. In particular we will show that a "normal form" (a fixed strategy) for successful derivations exists.

We need the following terminology and notation.

Let  $G = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$  be a special 1S grammar.

If  $A \in \Sigma \setminus \Delta$  is such that  $\text{alph}(\text{contr}_G(A)) \cap B_G \neq \emptyset$  then we call  $A$  a central symbol (of  $G$ ). By  $C(G)$  we denote the set of central symbols of  $G$ .

Let  $u$  be a sentential form of  $G$ . Then  $u$  contains at most one occurrence of a central symbol. If  $u$  contains exactly one occurrence of a central symbol, then we say that  $u$  is a central sentential form (of  $G$ ).

If  $u$  is a central sentential form of  $G$  then we denote by  $\text{cent}_G(u)$  the unique central symbol in  $u$  and by  $\text{Occ}_G(u)$  its position in  $u$ . Hence if  $u = u_1 A u_2$ , for some  $u_1, u_2 \in (\Sigma \setminus C(G))^*$  and  $A \in C(G)$ , then  $\text{cent}_G(u) = A$  and  $\text{Occ}_G(u) = |u_1 A|$ . Whenever  $G$  is clear from the context we omit the subscript  $G$ .

Since  $G$  is special all non-terminals of  $G$  are either in  $L_G$  or in  $R_G$  (or in both). Hence to rewrite a successful sentential form  $u$  one has at most two options:

Either one rewrites the right-most left-blocking non-terminal in  $u$  (if any) or one rewrites the left-most right-blocking non-terminal in  $u$  (if any).

Notice that since either  $\text{Locc}(u) \geq 1$  or  $\text{Rocc}(u) \leq |u|$ , at least one of these situations occurs and that it may be that  $\text{Locc}(u) = \text{Rocc}(u)$  (a blocking symbol).

We will describe a strategy, how to make a choice between  $\text{Locc}(u)$  and  $\text{Rocc}(u)$ , in the case one has two options. This strategy depends on the central symbol in the current sentential form.

Observe that whenever  $u$  is a non-central sentential form of  $G$ , then there exists no  $v \in \Sigma^*$  such that  $u \Rightarrow_G v$  ( $u$  is blocked).



Definition 3.1. (1). Let  $u$  be a central sentential form of  $G$ . Then the designated occurrence in  $u$ , denoted  $\text{Des}_G(u)$ , is defined by

$$\text{Des}_G(u) = \begin{cases} \text{Locc}_G(u), & \text{if } \text{Rocc}_G(u) > |u|, \\ \text{Locc}_G(u), & \text{if } 1 \leq \text{Locc}_G(u) \leq \text{Rocc}_G(u) \leq \text{Occ}_G(u) \leq |u|, \\ \text{Rocc}_G(u), & \text{if } \text{Locc}_G(u) < 1, \\ \text{Rocc}_G(u), & \text{if } 1 \leq \text{Occ}_G(u) \leq \text{Locc}_G(u) \leq \text{Rocc}_G(u) \leq |u|. \end{cases}$$

(2). Let  $D = (x_0, x_1, \dots, x_n)$ ,  $n \geq 0$ , be a derivation in  $G$ . For  $0 \leq i \leq n-1$ , the derivation step  $(x_i, x_{i+1})$  is central if  $\text{Rew}(D, i) = \text{Des}_G(x_i)$ .  $D$  is a central derivation (in  $G$ ) if, for all  $0 \leq i \leq n-1$ ,  $(x_i, x_{i+1})$  is central.  $\square$

We will write  $\text{Des}(u)$  rather than  $\text{Des}_G(u)$ , whenever  $G$  is clear from the context.

Hence in a central derivation step the designated occurrence is rewritten, i.e.

- (1). the right-most left-blocking non-terminal if  
 (1.i) no right-blocking non-terminals are present, or  
 (1.ii) the central symbol is right-blocking;

- (2). the left-most right-blocking non-terminal if  
 (2.i) no left-blocking non-terminals are present, or  
 (2.ii) the central symbol is left-blocking.

Observe that if the sentential form contains a blocking non-terminal, then the right-most left-blocking and left-most right-blocking non-terminals are the same.

That the central derivations are derivations in a "normal form" and that one can consider central derivations only in special 1S grammars is proved as follows.

Theorem 3.1. Let  $G$  be a special 1S grammar. For every  $w \in L(G)$  there exists a central derivation of  $w$  in  $G$ .

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Proof. Let  $G = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$ . For a derivation  $D$  in  $G$ ,  $v(D)$  denotes the number of non-central derivation steps in  $D$ .

Let  $w \in L(G)$  and let  $D_w$  be a derivation of  $w$  such that  $v(D_w) = \min \{v(D) : D \text{ is a derivation of } w \text{ in } G\}$ . We will prove by contradiction that  $v(D_w) = 0$ . Hence, we assume that  $v(D_w) > 0$ .

Let  $D_w = (x_0, x_1, \dots, x_n)$ , where  $x_i \in \Sigma^*$  for  $1 \leq i \leq n$ ,  $x_0 = S$  and  $x_n = w$ . Let  $i_0 \in \{1, \dots, n-1\}$  be the smallest  $i$  such that the derivation step  $(x_i, x_{i+1})$  in  $D$  is non-central.

If  $\underline{\text{cent}}(x_{i_0}) \in B_G$ , then  $\text{cent}(x_{i_0})$  is rewritten in  $(x_{i_0}, x_{i_0+1})$ . This would imply that  $(x_{i_0}, x_{i_0+1})$  is central. A contradiction. Hence  $\underline{\text{cent}}(x_{i_0}) \in X \cup Z$  and since  $G$  is special, this implies that  $\underline{\text{cent}}(x_{i_0}) \in (X \setminus Z) \cup (Z \setminus X)$ .

Let us assume that  $\underline{\text{cent}}(x_{i_0}) \in X \setminus Z$  (the other case is symmetric and can be dealt with analogously). Since  $\underline{\text{cent}}(x_{i_0}) \in L_G \setminus R_G$  and  $(x_{i_0}, x_{i_0+1})$  is non-central, it follows that  $\text{Rocc}(x_{i_0}) \leq |x_{i_0}|$ ,  $\text{Locc}(x_{i_0}) \neq \text{Rocc}(x_{i_0})$  and  $\text{Rew}(D, i_0) = \text{Locc}(x_{i_0})$ . Let  $r = \text{Rocc}(x_{i_0})$  and  $k = |x_{i_0}|$ ; hence  $1 < r \leq k$ .

Let  $\alpha_0 = a_1 \dots a_{r-1}$  and  $\beta_0 = a_r \dots a_k$ , where  $a_i \in \Sigma$ , for  $1 \leq i \leq k$ , and such that  $x_{i_0} = \alpha_0 \beta_0$ .

Hence  $D_w = (x_0, x_1, \dots, x_{i_0-1}, \alpha_0 \beta_0, \alpha_1 \beta_1, \dots, \alpha_q \beta_q)$ , where  $q = n - i_0$  and, for  $0 \leq i \leq q$ ,  $\alpha_i \beta_i = x_{i_0+i}$  and for  $0 \leq i \leq q-1$ :

- (1) if  $\text{Rew}(D_w, i_0+i) > |\alpha_i|$ , then  $\alpha_i = \alpha_{i+1}$  and  $\beta_i = \beta_{i+1}$ , and
- (2) if  $\text{Rew}(D_w, i_0+i) \leq |\alpha_i|$ , then  $\alpha_i = \alpha_{i+1}$  and  $\beta_i = \beta_{i+1}$ .

Let  $I = \{i : 0 \leq i \leq q-1 \text{ and } \text{Rew}(D_w, i_0+i) > |\alpha_i|\}$  and  $J = \{i : 0 \leq i \leq q-1 \text{ and } \text{Rew}(D_w, i_0+i) \leq |\alpha_i|\}$ . Note that  $I \cap J = \emptyset$ .

Let  $i_1, \dots, i_p$  be the elements of  $I$  in ascending order and let  $j_1, \dots, j_s$  be the elements of  $J$  in ascending order. Hence  $p, s \geq 1$  and  $p+s = q$ ,

$\alpha_{j_1} = \alpha_0$ ,  $\beta_{i_1} = \beta_0$ ,  $\alpha_{j_i} = \alpha_{j_i+1}$ , for  $1 \leq i \leq s-1$ ,  $\alpha_{j_s} = \alpha_q$  and  $\beta_{i_j} = \beta_{i_j+1}$ , for  $0 \leq j \leq p-1$  and  $\beta_{i_p} = \beta_q$ .

We set  $D' = (x_0, x_1, \dots, x_{i_0-1}, \alpha_0 \beta_{i_1}, \dots, \alpha_0 \beta_{i_p}, \alpha_0 \beta_q, \alpha_{j_2} \beta_q, \dots, \alpha_{j_s} \beta_q, \alpha_q \beta_q)$ . We claim that  $D'$  is a derivation of  $w$  in  $G$  and  $v(D') \leq v(D_w) - 1$ .

The situation can be illustrated as follows.

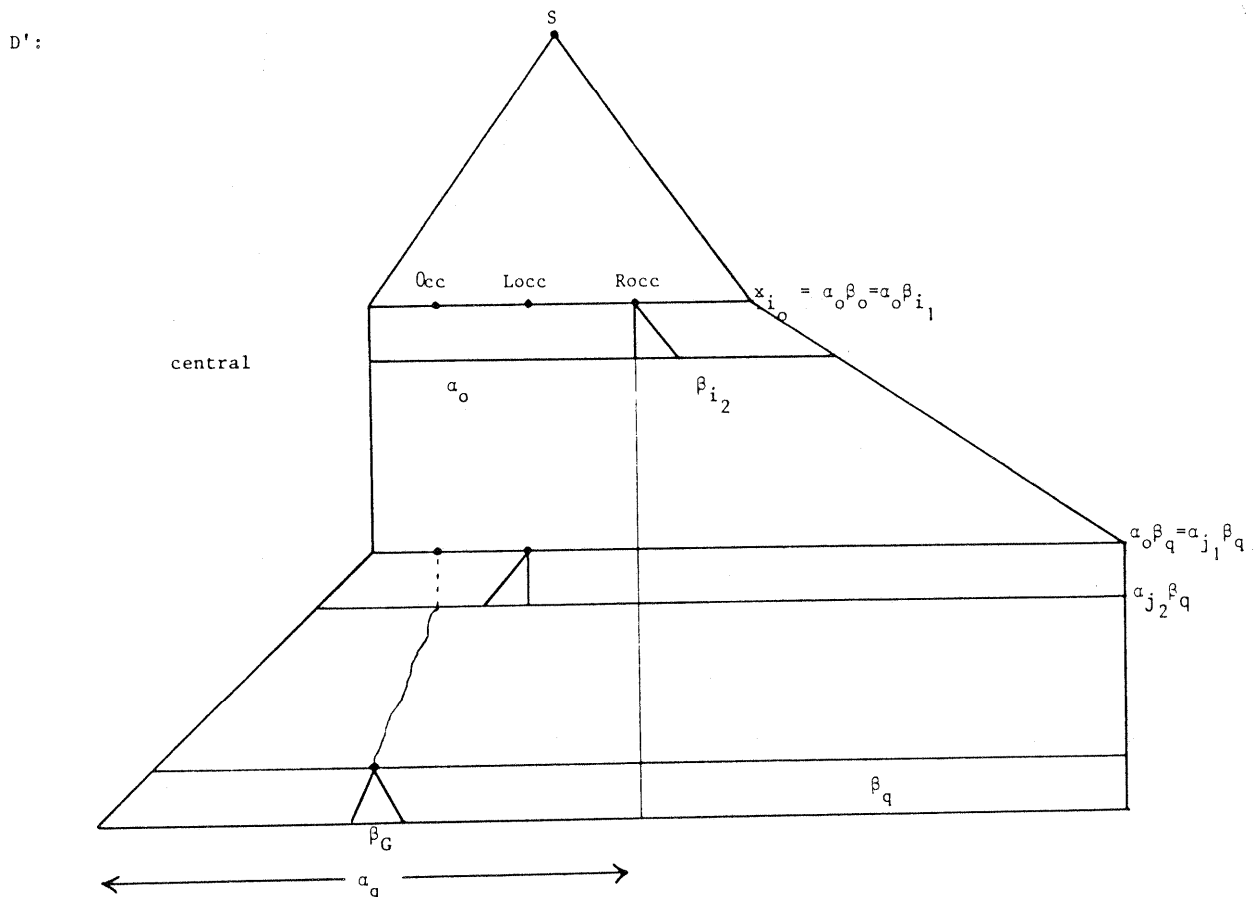
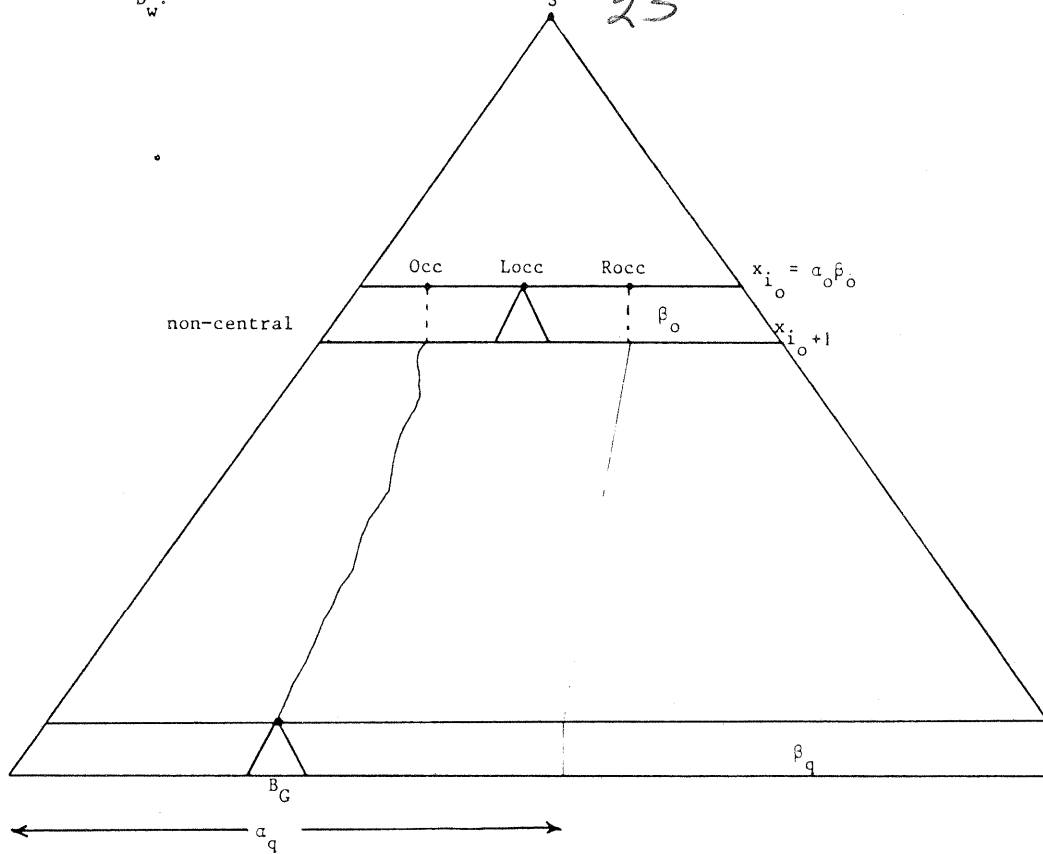


Figure 3.1.

First we recall that  $\alpha_q \beta_q = w$ .

Next we consider  $D_w$ .

$\text{Rocc}(x_{i_0}) = \text{Rocc}(\alpha_0 \beta_0) = |\alpha_0| + 1$ . Hence  $\alpha_0 \in X^{**}$ . This implies that, for  $0 \leq j \leq p-1$ ,  $\alpha_0 \beta_{i_j} = \alpha_0 \beta_{i_{j+1}}$  and  $\alpha_0 \beta_{i_p} = \alpha_0 \beta_q$ . Recall furthermore that  $\text{Occ}(\alpha_0 \beta_0) \leq \text{Locc}(\alpha_0 \beta_0) \leq |\alpha_0|$ . Since  $G$  is special this implies that  $\text{contr}_G(\beta_0) \in Z^{**}$  and in particular that  $\beta_q \in Z^{**}$ . Hence, for  $0 \leq i \leq s-1$ ,  $\alpha_{j_i} \beta_q = \alpha_{j_{i+1}} \beta_q$  and  $\alpha_{j_s} \beta_q = \alpha_q \beta_q$ .

This proves that  $D'$  is a derivation of  $w$  in  $G$ .

It remains to show that  $v(D') \leq v(D_w) - 1$ .

(1) Consider the step  $(\alpha_0 \beta_{i_1}, \alpha_0 \beta_{i_2})$  in  $D'$ . Since  $\alpha_0 \beta_{i_1} = \alpha_0 \beta_0$  and  $\text{Rocc}(\alpha_0 \beta_0) \geq |\alpha_0| + 1$ , all non-terminals in  $\beta_0$  are right-blocking. Hence it is the first non-terminal symbol of  $\beta_0$  that is rewritten in this step. Since  $\text{Occ}(\alpha_0 \beta_0) \leq \text{Locc}(\alpha_0 \beta_0) \leq |\alpha_0|$ , it follows that this step in  $D'$  is central.

(2) Consider a step  $(\alpha_0 \beta_{i_j}, \alpha_0 \beta_{i_{j+1}})$  in  $D'$ , for some  $2 \leq j \leq p$  where  $\beta_{i_{p+1}} = \beta_q$ , and assume that this step is non-central. We will show that the step  $(\alpha_{i_j} \beta_{i_j}, \alpha_{i_{j+1}} \beta_{i_{j+1}})$  in  $D_w$  is also non-central. Since  $\text{cent}(\alpha_0 \beta_{i_j}) \in L_G$ , it follows that  $\text{Occ}(\alpha_0 \beta_{i_j}) \leq |\alpha_0|$ . Since the rewriting takes place in  $\beta_{i_j}$  and the step under consideration is non-central, it follows that  $\text{Locc}(\alpha_0 \beta_{i_j}) > |\alpha_0|$ ,  $\text{Rocc}(\alpha_0 \beta_{i_j}) \leq |\alpha_0 \beta_{i_j}|$  and the right-most left-blocking symbol is rewritten. Hence  $\text{Locc}(\alpha_{i_j} \beta_{i_j}) > |\alpha_{i_j}|$ ,  $\text{Rocc}(\alpha_{i_j} \beta_{i_j}) \leq |\alpha_{i_j} \beta_{i_j}|$  and in the step  $(\alpha_{i_j} \beta_{i_j}, \alpha_{i_{j+1}} \beta_{i_{j+1}})$  in  $D_w$  the right-most left-blocking symbol is rewritten.

This implies that this step is not central in  $D_w$ .

(3) Consider a step  $(\alpha_{j_i} \beta_q, \alpha_{j_{i+1}} \beta_q)$  in  $D'$ , for some  $1 \leq i \leq s$ , where  $\alpha_{j_{s+1}} = \alpha_q$  and assume that this step is non-central. We will show that the step  $(\alpha_{j_i} \beta_{j_i}, \alpha_{j_{i+1}} \beta_{j_{i+1}})$  in  $D_w$  is also non-central.

Since  $\beta_q \in \Delta^{**}$  it follows that if  $\text{cent}(\alpha_{j_i} \beta_q) \in L_G$  then  $\text{Rocc}(\alpha_{j_i} \beta_q) \leq |\alpha_{j_i}|$  and the right-most left-blocking symbol in  $\alpha_{j_i}$  is rewritten, and if

$\text{cent}(\alpha_{j_i} \beta_q) \in R_G$  then  $\text{Locc}(\alpha_{j_i} \beta_q) \geq 1$  and the left-most right-blocking symbol in  $\alpha_{j_i}$  is rewritten in  $(\alpha_{j_i} \beta_q, \alpha_{j_{i+1}} \beta_q)$ .

Then  $(\alpha_{j_i} \beta_{j_i}, \alpha_{j_{i+1}} \beta_{j_{i+1}})$  is non-central, because  $\text{Rew}(D_w, j_i) = \text{Rew}(D', i_0 + p + i)$ .

(1), (2) and (3) together imply that the number of non-central steps in the first  $(i_0 + 1)$  derivation steps of  $D_w$  and  $D'$  are 1 and 0 respectively; in the last  $n - i_0 - 1$  steps each non-central step in  $D'$  corresponds to a distinct non-central step in  $D_w$ . Hence  $v(D') \leq v(D_w) - 1$ , which contradicts the minimality of  $v(D_w)$ .

Thus for every  $w \in L(G)$  there exists a central derivation of  $w$  in  $L(G)$  which proves the theorem. □

4. SPECIAL IS GRAMMARS GENERATE CONTEXT-FREE LANGUAGES

In this section we consider an arbitrary but fixed, special IS grammar  $G = (\Sigma, P, S, \Delta, X^* \bar{Y} Z^*)$  and its central derivations. We will show that only a "finite memory" is needed to prevent unsuccessfulness (blocking) of central derivations. By building in this finite memory an equivalent special IS grammar is constructed from  $G$ , in which all central derivations are successful. Then it can be shown that  $L(G)$  is context-free.

First we identify the situations in which a central derivation gets blocked.

Let  $(A, \alpha) \in P$ , for some  $A \in \Sigma \setminus \Delta$  and  $\alpha \in \Sigma^*$ .

$(A, \alpha)$  is a left-blocking production if  $\text{alph}(\alpha) \cap (\Delta \setminus Z) \neq \emptyset$ ,

$(A, \alpha)$  is a right-blocking production if  $\text{alph}(\alpha) \cap (\Delta \setminus X) \neq \emptyset$  and

$(A, \alpha)$  is a blocking production if it is both left- and right-blocking.

Observe that whenever a left-(right-) blocking production is used to rewrite a sentential form, then the non-terminals that occur to the left (right) of the rewritten occurrence cannot be rewritten anymore.

Hence in order not to block a central derivation, whenever a left-(right-) blocking production is applied to a sentential form  $w$ , the left (right) description of  $w$  should be terminal. We define two functions on the set of central sentential forms of  $G$ , that describe the left and the right context of the designated occurrences in sentential forms. These functions ld and rd respectively have as their codomain the set  $\{\underline{t}, \underline{nt}\}$ , where t stands for terminal and nt for non-terminal.

Let  $w$  be a central sentential form of  $G$ . Let  $u, v \in \Sigma^*$  and  $A \in \Sigma \setminus \Delta$  be such that  $w = uAv$  and  $\text{Des}(w) = |uA|$ . Then

$$\begin{aligned} \underline{ld}(w) &= \begin{cases} \underline{t} & \text{if } u \in \Delta^*, \\ \underline{nt} & \text{otherwise,} \end{cases} \\ \underline{rd}(w) &= \begin{cases} \underline{t} & \text{if } v \in \Delta^*, \\ \underline{nt} & \text{otherwise.} \end{cases} \end{aligned}$$

Let  $D = (x_0, x_1, \dots, x_n)$  be a central derivation in  $G$ . For  $0 \leq k \leq n-1$ , we call  $(x_k, x_{k+1})$  careful if whenever  $\text{Prod}(D, k)$  is left-blocking, then  $\underline{ld}(x_k) = \underline{t}$  and whenever  $\text{Prod}(D, k)$  is right-blocking, then  $\underline{rd}(x_k) = \underline{t}$ .  $D$  is careful if, for all  $0 \leq k \leq n-1$ ,  $(x_k, x_{k+1})$  is careful.

Lemma 4.1. Let  $D = (x_0, x_1, \dots, x_n)$  be a central derivation. If  $x_n \in \Delta^*$ , then  $D$  is careful.

Proof. Obvious.  $\square$

Hence if a central derivation is not careful it will never "lead to success".

On the other hand all central derivations that get blocked (in a non-trivial way), are not careful. This is shown in the following lemma.

Lemma 4.2. Let  $D = (x_0, x_1, \dots, x_n)$ ,  $n \geq 0$ , be a central derivation in  $G$ , such that  $x_n \notin \Delta^*$ . If there exists no  $y$  such that  $(x_0, x_1, \dots, x_n, y)$  is a central derivation then either

- (1)  $n = 0$  and  $P = \emptyset$ , or
- (2)  $n \geq 1$  and  $D$  is not careful.

Proof. (1)  $n = 0$ .

In this case  $D = (S)$ .  $G$  is strongly reduced. Hence, if

$P \neq \emptyset$ , then  $P \cap (S \times \Sigma^*) \neq \emptyset$ . Since  $S \in Y$ , this implies that  $S \stackrel{G}{=} \alpha$ , for some  $\alpha \in \Sigma^*$  and  $(S, \alpha) \in P$ . Then, obviously,  $(S, \alpha)$  is a central derivation.

This is a contradiction and so  $P = \emptyset$ .

(2)  $n \geq 1$ .

Hence  $P \neq \emptyset$  and for all non-terminal symbols productions are available.

We distinguish the following two cases.

(2.i) There exists no  $y$  such that  $x_n \stackrel{G}{=} y$ .

Consider the central derivation step  $(x_{n-1}, x_n)$  in  $D$ . Let  $x_{n-1} = xAz$ , with  $|xA| = \text{Des}(x_{n-1})$ ,  $x_n = x\alpha z$ , for some  $x \in X^*$ ,  $z \in Z^*$  and  $(A, \alpha) \in P$ .  $(A, \alpha)$  is a safe production. Hence if  $\alpha \notin \Delta^*$ , then  $\alpha \in X^*YZ^*$  and so  $x_n \in X^*YZ^*$ . This implies that there exists a  $y \in \Sigma^*$  such that  $x_n \xrightarrow{G} y$ . This contradicts our assumption. Hence  $\alpha \in \Delta^*$ . If  $A \notin C(G)$ , then either  $\text{Occ}(x_{n-1}) \leq |x|$  or  $\text{Occ}(x_{n-1}) > |xA|$ . If  $\text{Occ}(x_{n-1}) \leq |x|$ , then  $\alpha \in Z^*$  (condition (C5) from Definition 2.5). Moreover,  $x = uBv$ , for some  $u \in X^*$ ,  $B \in \Sigma \setminus \Delta$  and  $v \in (X \cap \Delta)^*$ . Since  $\text{Occ}(x_{n-1}) \leq |uB|$ ,  $v \in (Z \cap X \cap \Delta)^*$ . Hence  $x_n = x\alpha z = uBv\alpha z$ , with  $u \in X^*$  and  $v\alpha z \in Z^*$ . This implies that there exists a  $y \in \Sigma^*$  such that  $x_n \xrightarrow{G} y$ . This contradicts our assumption. Similarly one can prove that the case  $\text{Occ}(x_{n-1}) > |xA|$  leads to a contradiction. Hence  $A \in C(G)$ . This implies that  $\alpha \in (B_G \cap \Delta)$  and  $(A, \alpha)$  is a blocking production. Since  $x_n \notin \Delta^*$ , either  $\underline{\text{ld}}(x_{n-1}) = \underline{\text{nt}}$  or  $\underline{\text{rd}}(x_{n-1}) = \underline{\text{nt}}$ . Hence  $(x_{n-1}, x_n)$  is not a careful derivation step and so  $D$  is not careful.

(2.ii) There exists a  $y$  such that  $x_n \xrightarrow{G} y$ .

Hence  $x_n$  is a central sentential form. Since  $x_n$  can not be rewritten in the central way,  $x_n$  contains a left-blocking non-terminal and a right-blocking non-terminal, which are not the same. (Otherwise there is only one way of rewriting  $x$ , which then is automatically the central way.) Hence  $1 \leq \text{Locc}(x_n) < \text{Rocc}(x_n) \leq |x_n|$  and either

(I)  $\text{Des}(x_n) = \text{Locc}(x_n)$  and  $\underline{\text{cent}}(x_n) \in R_G$  or

(II)  $\text{Des}(x_n) = \text{Rocc}(x_n)$  and  $\underline{\text{cent}}(x_n) \in L_G$ .

We consider case (I).

In this case  $x_n = \alpha A \beta B \gamma$ , with  $\alpha, \gamma \in \Sigma^*$ ,  $\beta \in \Delta^*$  and  $A, B \in \Sigma \setminus \Delta$ , and  $\text{Locc}(x_n) = |\alpha A|$  and  $\text{Rocc}(x_n) = |\alpha A \beta B|$ . Since  $x_n$  can be rewritten, but not in the central way,  $\alpha A \beta \in X^*$  and  $\gamma \in Z^*$ . Moreover,  $B \in Z \setminus X$  and so  $\beta \in X^* \setminus Z^*$ , i.e.  $\beta = \beta_1 b \beta_2$ , for some  $\beta_1 \in \Delta^*$ ,  $\beta_2 \in (\Delta \cap Z)^*$  and  $b \in \Delta \setminus Z$ . Observe, that since all productions are safe, this occurrence of  $b$  is introduced using a production of the form  $(A', ubv)$  where  $u \in \Delta^*$  and  $v \in (\Sigma \setminus \{b\})^*$ .



Let  $j = \max\{i: 0 \leq i \leq n-1, \text{Rew}(D,i) \leq |\alpha A \beta_1 b|\}$ . Hence  $x_j = \alpha A \beta_3 A' \delta$  and  $\text{Prod}(D,j) = (A'ubv)$ , as above, where  $\beta_3 u = \beta_1$  and  $\delta \in \Sigma^*$ .

Hence  $\text{Prod}(D,j)$  is left-blocking and  $\underline{\text{ld}}(x_j) = \underline{\text{nt}}$ .

This implies that  $(x_j, x_{j+1})$  is not careful.

Case II can be treated analogously.  $\square$

Consequently, central derivations can be prolonged in the central way as long as the designated occurrences "know" about their context and "act" accordingly.

We will show that only a finite memory is needed for this knowledge.

We need the following notions.

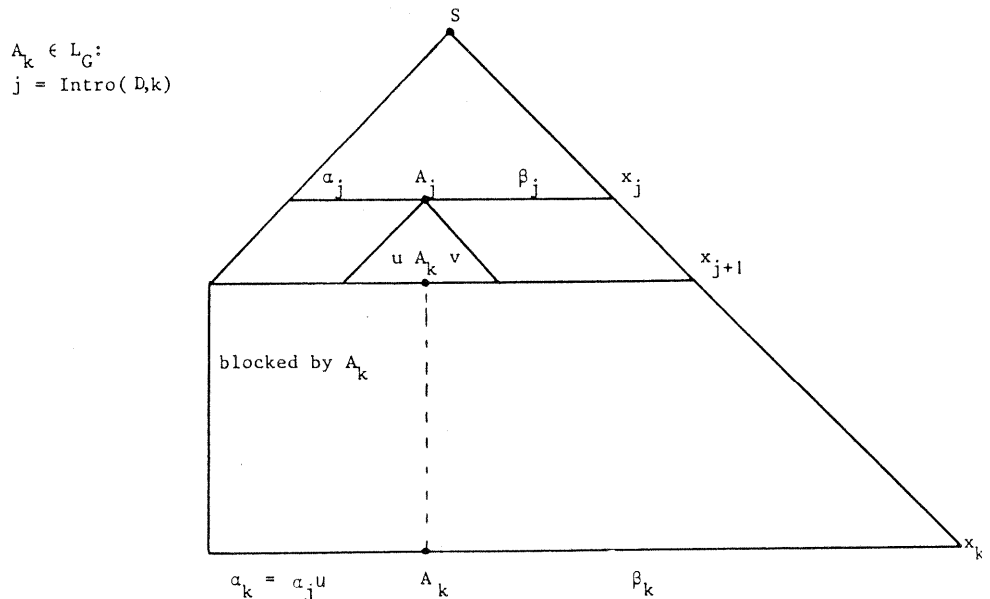
Let  $D = (x_0, x_1, \dots, x_n)$ ,  $n \geq 0$ , be a central derivation in  $G$ , such that  $x_n$  is a central sentential form. Let, for  $1 \leq k \leq n$ ,  $x_k = \alpha_k A_k \beta_k$ , for some  $\alpha_k, \beta_k \in \Sigma^*$  and  $A_k \in \Sigma \setminus \Delta$ , where  $|\alpha_k A_k| = \underline{\text{Des}}(x_k)$ .

Then, for  $1 \leq k \leq n$ ,

$$\text{Intro}(D,k) = \begin{cases} \max\{j : 0 \leq j \leq k-1, |\alpha_j| \leq |\alpha_k|\} & \text{if } A_k \in L_G, \\ \max\{j : 0 \leq j \leq k-1, |\beta_j| \leq |\beta_k|\} & \text{if } A_k \in R_G. \end{cases}$$

Observe that  $\text{Intro}(D,k)$  describes in which derivation step in  $D$  the designated occurrence in  $x_k$  has been introduced.

This can be illustrated as follows.



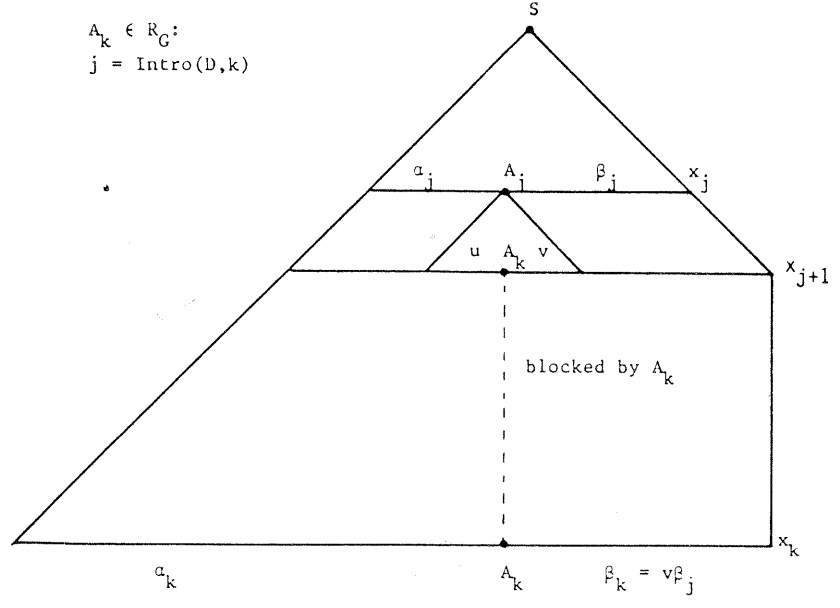


Figure 4.1.

Let  $k$  and  $j$  be such that  $j = \text{Intro}(D,k)$ ,  $0 \leq j < k \leq n$ . Let  $u, v \in \Sigma^*$  be such that  $\text{Prod}(D,j) = (A_j, uA_kv)$  and  $\alpha_k = \alpha_j u$  if  $A_k \in L_G$  and  $\beta_k = v\beta_j$  if  $A_k \in R_G$ . Then prefintro  $(D,k) = u$  and postintro  $(D,k) = v$ .

Hence prefintro  $(D,k)$  and postintro  $(D,k)$  describe with what left and right, respectively, context the designated occurrence in  $x_k$  has been introduced.

If  $\text{Des}(x_k) < \text{Occ}(x_k)$ , for some  $0 \leq k \leq n$ , then  $\underline{rd}(x_k) = \underline{nt}$  and, since  $G$  is special, the step  $(x_k, x_{k+1})$  will be careful (Condition (C5) of Definition 2.5).

Analogously, if  $\text{Des}(x_k) > \text{Occ}(x_k)$ , then  $\underline{ld}(x_k) = \underline{nt}$  and  $(x_k, x_{k+1})$  is careful.

The following lemma shows that for the remaining situations it suffices to know about the introduction of  $\text{Des}(x_k)$ .

Lemma 4.3. There exist functions  $F_L$  and  $F_R$ , such that for every central derivation  $D = (x_0, x_1, \dots, x_n)$  in  $G$ , for every  $i, j \in \{0, \dots, n-1\}$  such that  $i = \text{Intro}(D,j)$ ,  $x_j = \alpha A \beta$ , for some  $\alpha, \beta \in \Sigma^*$  and  $A \in \Sigma \setminus \Delta$ , where  $\text{Des}(x_j) = |\alpha A|$ ,

$\underline{\ell}d(x_j) = F_L(\underline{\ell}d(x_i), \underline{\text{prefintro}}(D, j), A)$  if  $\text{Des}(x_j) \leq \text{Occ}(x_j)$ , and  
 $\underline{r}d(x_j) = F_R(\underline{r}d(x_i), \underline{\text{postintro}}(D, j), A)$  if  $\text{Des}(x_j) \geq \text{Occ}(x_j)$ .

Proof. Let  $F_L$  and  $F_R$  be defined in the following way.

For  $x \in \{\underline{t}, \underline{nt}\}$ ,  $w \in \Sigma^*$  and  $E \in \Sigma \setminus \Delta$ ,

$$F_L(x, w, E) = \begin{cases} \underline{t} & \text{if } E \in R_G \setminus L_G, \\ \underline{t} & \text{if } E \in L_G, x = \underline{t} \text{ and } w \in \Delta^*, \\ \underline{nt} & \text{otherwise,} \end{cases}$$

and

$$F_R(x, w, E) = \begin{cases} \underline{t} & \text{if } E \in L_G \setminus R_G, \\ \underline{t} & \text{if } E \in R_G, x = \underline{t} \text{ and } w \in \Delta^*, \\ \underline{nt} & \text{otherwise.} \end{cases}$$

Let  $D, i, j$  and  $x_j$  be as in the statement of the lemma and let  $\text{Prod}(D, i) = (A', uAv)$ , for some  $u, v \in \Sigma^*$  and  $A' \in \Sigma \setminus \Delta$ , such that  $u = \underline{\text{prefintro}}(D, j)$  and  $v = \underline{\text{postintro}}(D, j)$ . Hence  $x_j = \gamma A' \delta$  and  $x_{j+1} = \gamma u A v \delta$  for some  $\gamma, \delta \in \Sigma^*$ .

Moreover

if  $A \in L_G \setminus R_G$ , then  $\gamma u = \alpha$

if  $A \in R_G \setminus L_G$ , then  $v \delta = \beta$ , and

if  $A \in L_G \cap R_G$ , then  $j = i+1$ ,  $\gamma u = \alpha$  and  $v \delta = \beta$ .

If  $\text{Des}(x_j) \leq \text{Occ}(x_j)$ , then

if  $A \in R_G \setminus L_G$ , then  $\text{Locc}(x_j) = 0$  and so  $\alpha \in \Delta^*$ .

If  $\text{Des}(x_j) \geq \text{Occ}(x_j)$ , then

if  $A \in L_G \setminus R_G$ , then  $\text{Rocc}(x_j) = |x_j| + 1$  and so  $\beta \in \Delta^*$ .

Hence if  $\text{Des}(x_j) \leq \text{Occ}(x_j)$ , then

$$\underline{\ell}d(x_j) = \begin{cases} \underline{t} & \text{if } A \in R_G \setminus L_G, \\ \underline{t} & \text{if } A \in L_G, \gamma \in \Delta^* \text{ and } u \in \Delta^*, \\ \underline{nt} & \text{otherwise,} \end{cases}$$

hence  $\underline{\ell}d(x_j) = F_L(\underline{\ell}d(x_i), u, A)$ .

Similarly, it follows that if  $\text{Des}(x_j) \geq \text{Occ}(x_j)$ , then  $\underline{r}d(x_j) = F_R(\underline{r}d(x_i), v, A)$ .  $\square$

From  $G$  we construct the 1S grammar  $G_1 = (\Sigma_1, P_1, S_1, \Delta_1, X_1^*, \bar{Y}_1, Z_1^*)$  in the following way.

$\Sigma_1 = \Delta \cup [\cdot, \Gamma] \cup [\cdot, \Gamma, \cdot] \cup [\Gamma, \cdot]$ . Here  $[\cdot, \Gamma], [\cdot, \Gamma, \cdot]$  and  $[\Gamma, \cdot]$  are mutually disjoint alphabets also disjoint with  $\Sigma$ , defined as follows. For  $x, y \in \{\underline{t}, \underline{nt}\}$ ,

$[x, \Gamma] = \{[x, A] : A \in (\Sigma \setminus \Delta) \setminus C(G)\}$ ,  $[x, \Gamma, y] = \{[x, A, y] : A \in C(G)\}$  and  $[\Gamma, y] = \{[A, y] : A \in (\Sigma \setminus \Delta) \setminus C(G)\}$  are mutually disjoint and  $[\cdot, \Gamma] = [\underline{t}, \Gamma] \cup [\underline{nt}, \Gamma]$ ,

$$[\cdot, \Gamma, \cdot] = \bigcup_{\substack{x, y \in \\ \{\underline{t}, \underline{nt}\}}} [x, \Gamma, y] \text{ and}$$

$$[\Gamma, \cdot] = [\Gamma, \underline{t}] \cup [\Gamma, \underline{nt}].$$

$$S_1 = [\underline{t}, S, \underline{t}] \text{ and } \Delta_1 = \Delta.$$

$P_1$  is defined in the following way. For  $\pi = (A, \alpha) \in P$  with  $A \notin C(G)$  and

$$\alpha = \alpha_1 A_1 \alpha_2 \dots \alpha_n A_n \alpha_{n+1}, \text{ for some } n \geq 0, \alpha_1, \dots, \alpha_{n+1} \in \Delta^* \text{ and } A_1, \dots, A_n \in \Sigma \setminus \Delta,$$

we define  $[x, \pi]$  and  $[\pi, x]$ ,  $x \in \{\underline{t}, \underline{nt}\}$ , as follows.

$$[x, \pi] = \emptyset \text{ if } x = \underline{nt} \text{ and } \alpha_1 \dots \alpha_{n+1} \notin Z^*,$$

$$[\pi, x] = \emptyset \text{ if } x = \underline{nt} \text{ and } \alpha_1 \dots \alpha_{n+1} \notin X^*,$$

otherwise  $[x, \pi] = \{([x, A], \alpha_1 [x_1, A_1] \alpha_2 \dots \alpha_n [x_n, A_n] \alpha_{n+1}) : \text{for } 1 \leq i \leq n,$

$$x_i = F_L(x, \alpha_1 A_1 \dots \alpha_{i-1} A_{i-1} \alpha_i, A_i)\}$$

and

$$[\pi, x] = \{([A, x], \alpha_1 [A_1, x_1] \alpha_2 \dots \alpha_n [A_n, x_n] \alpha_{n+1}) : \text{for } 1 \leq i \leq n, x_i =$$

$$F_R(x, \alpha_{i+1} A_{i+1} \dots \alpha_n A_n \alpha_{n+1}, A_n)\}.$$

For  $\pi = (A, \alpha) \in P$ , with  $A \in C(G)$  and  $\alpha \in \Delta \cap B_G$ , we define  $[x, \pi, y]$ ,  $x, y \in \{\underline{t}, \underline{nt}\}$ , as follows.

$$[x, \pi, y] = \emptyset \text{ if } x = \underline{nt} \text{ or } y = \underline{nt}, \text{ and}$$

$$[\underline{t}, \pi, \underline{t}] = \{([\underline{t}, A, \underline{t}], \alpha)\}.$$

For  $\pi = (A, \alpha \beta) \in P$ , with  $A, C \in C(G)$ ,  $\alpha = \alpha_1 A_1 \alpha_2 \dots \alpha_n A_n \alpha_{n+1}$  and  $\beta = \beta_1 B_1 \beta_2 \dots$

$\beta_m B_m \beta_{m+1}$ , for some  $n, m \geq 0$ ,  $\alpha_1, \dots, \alpha_{n+1}, \beta_1, \dots, \beta_{m+1} \in \Delta^*$ ,

$A_1, \dots, A_n, B_1, \dots, B_m \in (\Sigma \setminus \Delta) \setminus C(G)$ , we define  $[x, \pi, y]$ ,  $x, y \in \{\underline{t}, \underline{nt}\}$  as follows.

$[x, \pi, y] = \emptyset$  if  $x = \underline{nt}$  and  $\alpha_1 \dots \alpha_{n+1} \notin Z^*$ ,

$[x, \pi, y] = \emptyset$  if  $y = \underline{nt}$  and  $\beta_1 \dots \beta_{m+1} \notin X^*$ ,

otherwise  $[x, \pi, y] = \{([x, A, y], \alpha_1 [x_1, A_1] \alpha_2 \dots \alpha_n [x_n, A_n] \alpha_{n+1} [x_{n+1}, C, y_0]$

$\beta_1 [B_1, y_1] \beta_2 \dots \beta_m [B_m, y_m] \beta_{m+1})$ : for  $1 \leq i \leq n$ ,  $x_i = F_L(x, \alpha_1 A_1 \dots \alpha_{i-1} A_{i-1} \alpha_i, A_i)$ ,  
 $x_{n+1} = F_L(x, \alpha, C)$ ,  $y_0 = F_R(y, \beta, C)$ , for  $1 \leq i \leq m$ ,  $y_i = F_R(y, \beta_{i+1} B_{i+1} \dots \beta_m B_m \beta_{m+1}, B_i)$ .

Now we set  $P_1 = \bigcup_{x \in \{\underline{t}, \underline{nt}\}} \left( \bigcup_{\substack{(A, \alpha) \in P \\ A \notin C(G)}} ([x, (A, \alpha)] \cup [(A, \alpha), x]) \right)$   
 $\cup \left( \bigcup_{y \in \{\underline{t}, \underline{nt}\}} \bigcup_{\substack{(A, \alpha) \in P \\ A \in C(G)}} [x, (A, \alpha), y] \right)$ .

Let  $\mu \in \text{HOM}(\Sigma_1, \Sigma)$  be such that

$\mu(a) = a$ , for all  $a \in \Delta$ ,

$\mu([x, A]) = \mu([A, x]) = A$ , for all  $A \in (\Sigma \setminus \Delta) \setminus C(G)$  and  $x \in \{\underline{t}, \underline{nt}\}$ ,

$\mu([x, A, y]) = A$ , for all  $A \in C(G)$  and  $x, y \in \{\underline{t}, \underline{nt}\}$ .

$X_1 = \mu^{-1}(X)$ ,  $Y_1 = \mu^{-1}(Y)$  and  $Z_1 = \mu^{-1}(Z)$ .

Since  $G$  is special,  $G_1$  satisfies (C1) through (C5) from Definition 2.5, as can easily be checked.

Let  $G_2$  be the strongly reduced version of  $G_1$ . Then  $G_2$  satisfies all conditions from Definition 2.5 (cf. step(V) in the proof of Theorem 2.1) and hence  $G_2$  is special.

Lemma 4.4.  $L(G) \subseteq L(G_2)$ .

Proof. Let  $D = (x_0, x_1, \dots, x_n)$  be a successful central derivation in  $G$ . We show that there exist  $y_0, y_1, \dots, y_{n-1} \in \Sigma_1^*$  such that  $y_i \in \mu^{-1}(x_i)$ , for  $0 \leq i \leq n-1$  and  $D' = (y_0, y_1, \dots, y_{n-1}, x_n)$  is a successful derivation in  $G_1$ . Since  $G_2$  is the strongly reduced version of  $G_1$ ,  $D'$  is also a successful derivation in  $G_2$ .

Let, for  $0 \leq i \leq n-1$ ,  $x_i = u_i A_i v_i$ , for some  $u_i, v_i \in \Sigma^*$  and  $A_i \in \Sigma \setminus \Delta$ , such that  $\text{Rew}(D, i) = |u_i A_i|$ ;  $x_n = u_{n-1} a v_{n-1} \in \Delta^*$  with  $a \in B_G$ .

Let  $y_0 = [\underline{t}, S, \underline{t}] = S_1$ .

Assume that for some  $k \in \{0, \dots, n-1\}$ ,  $y_0, \dots, y_k$  have been defined in such a way that  $D'_k = (y_0, \dots, y_k)$  is a derivation in  $G_1$ , and, for  $0 \leq i \leq k$ ,  $\text{Des}(D'_k, i) = \text{Rew}(D, i)$ ,  $y_i = u'_i A'_i v'_i$ , with  $\mu(u'_i) = u_i$ ,  $\mu(A'_i) = A_i$  and  $\mu(v'_i) = v_i$  and

$A'_i = [x, A_i]$  if  $\text{Rew}(D, i) < \text{Occ}(x_i)$  with  $x = \underline{\text{ld}}(x_i)$ ,

$A'_i = [A_i, x]$  if  $\text{Rew}(D, i) > \text{Occ}(x_i)$  with  $x = \underline{\text{rd}}(x_i)$ , and

$A'_i = [x, A_i, y]$  if  $A_i = \underline{\text{cent}}(x_i)$ , with  $x = \underline{\text{ld}}(x_i)$  and  $y = \underline{\text{rd}}(x_i)$ .

Consider the step  $(x_k, x_{k+1})$ . Let  $\text{Prod}(D, k) = (A_k, \alpha)$ , for some  $\alpha \in \Sigma^*$ .

If  $k < n-1$ , then we proceed as follows.

$(x_k, x_{k+1})$  is a careful derivation step (Lemma 4.1.). Hence if  $\underline{\text{ld}}(x_k) = \underline{\text{nt}}$ , then  $\underline{\text{alph}}(\alpha) \cap (\Delta \setminus Z) = \emptyset$  and if  $\underline{\text{rd}}(x_k) = \underline{\text{nt}}$ , then  $\underline{\text{alph}}(\alpha) \cap (\Delta \setminus X) = \emptyset$ .

This implies that  $P_1$  contains a unique production  $(A'_k, \alpha')$  such that

$\mu(A'_k) = A_k$  and  $\mu(\alpha') = \alpha$ . Then we set  $y_{k+1} = u'_k \alpha' v'_k$ . Clearly  $\mu(y_k) = x_k$

and  $y_k = y_{k+1}$  with  $\text{Des}(D'_{k+1}, k) = \text{Rew}(D, k)$ .

Let  $u'_{k+1}, v'_{k+1}$  and  $A'_{k+1} \in \Sigma^*$  be such that  $\mu(u'_{k+1}) = u_{k+1}$ ,  $\mu(v'_{k+1}) = v_{k+1}$ ,

$\mu(A'_{k+1}) = A_{k+1}$  and  $y_{k+1} = u'_{k+1} A'_{k+1} v'_{k+1}$ .

Let  $j = \text{Intro}(D, k+1)$  and  $\text{Prod}(D, j) = (A'_j, \beta A'_{k+1} \gamma)$ , with  $\underline{\text{prefintro}}(D, k+1) = \beta$  and  $\underline{\text{postintro}}(D, k+1) = \gamma$ .

Since  $(x_j, x_{j+1})$  is careful,  $\text{Prod}(D'_k, j) = (A'_j, \beta' A'_{k+1} \gamma')$  with

$\mu(A'_j) = A_j$ ,  $\mu(\beta') = \beta$ ,  $\mu(A'_{k+1}) = A_{k+1}$  and  $\mu(\gamma') = \gamma$ , is unique. From the

construction of  $G_1$  it follows that

$A'_{k+1} = [\underline{\text{ld}}(x_{k+1}), A_{k+1}]$  if  $\text{Rew}(D, k+1) < \text{Occ}(x_{k+1})$ ,

$A'_{k+1} = [A_{k+1}, \underline{\text{rd}}(x_{k+1})]$  if  $\text{Rew}(D, k+1) > \text{Occ}(x_{k+1})$  and

$A'_{k+1} = [\underline{\text{ld}}(x_{k+1}), A_{k+1}, \underline{\text{rd}}(x_{k+1})]$  if  $A_{k+1} = \underline{\text{cent}}(x_{k+1})$ .

If  $k = n-1$ , then  $y_{n-1} = u_{n-1} [\underline{t}, A_{n-1}, \underline{t}] v_{n-1}$ , and  $\alpha = a$ .

$([\underline{t}, A_{n-1}, \underline{t}], a) \in P_1$  and so  $y_{n-1} \xrightarrow{G_1} x_n$ . Hence  $(y_0, \dots, y_{n-1}, x_n)$  is a successful derivation in  $G_1$ . □

Lemma 4.5. Let  $D = (x_0, x_1, \dots, x_n)$  be a central derivation in  $G_2$ . Then  $D_\mu = (\mu(x_0), \mu(x_1), \dots, \mu(x_n))$  is a careful central derivation in  $G$ .

Proof. Immediate. □

Theorem 4.1.  $L(G) = L(G_2)$ .

Proof. Immediate from Lemma 4.4. and Lemma 4.5. □

The following lemma shows that unless  $P_2 = \emptyset$ , no central derivations in  $G_2$  can get unsuccessfully blocked.

Lemma 4.6. Let  $D = (x_0, x_1, \dots, x_n)$  be a central derivation in  $G_2$  such that there exists no  $y \in \Sigma_2^*$  for which  $(x_0, x_1, \dots, x_n, y)$  is a central derivation in  $G_2$ .

Then either  $n = 0$  and  $P_2 = \emptyset$  or  $x_n \in \Delta_2^*$ .

Proof. (1).  $n = 0$ .

As in case (1) in the proof of Lemma 4.2 it follows that  $P_2 = \emptyset$ .

(2).  $n \geq 1$ . Hence  $P_2 \neq \emptyset$  and productions for all non-terminal symbols of  $G_2$  are available. From Lemma 4.5 it follows that  $(\mu(x_0), \mu(x_1), \dots, \mu(x_n))$  is a careful central derivation in  $G$ . From Lemma 4.2 it follows that if  $\mu(x_n) \notin \Delta^*$ , then there exists a  $v \in \Sigma^*$  such that  $(\mu(x_0), \mu(x_1), \dots, \mu(x_n), v)$  is a central derivation in  $G$ . Hence  $x_n = \alpha A \beta$ , for some  $\alpha \in X^*$ ,  $\beta \in Z^*$  and  $A \in \Sigma_2 \setminus \Delta_2$ , such that  $\text{Des}(x_n) = |\alpha A|$ . Then there exists a word  $y \in \Sigma_2^*$  such that  $(x_n, y)$  is a central derivation step in  $G_2$ . This is a contradiction. Hence  $\mu(x_n) \in \Delta^*$  which implies that  $\mu(x_n) = x_n \in \Delta_2^*$ . □

Lemma 4.7.  $L(G_2) = L(\text{base}(G_2))$ .

Proof. For every 1S grammar  $H$ ,  $L(H) \subseteq L(\underline{\text{base}}(H))$ . Hence  $L(G_2) \subseteq L(\underline{\text{base}}(G_2))$ .  
It thus remains to show that  $L(\underline{\text{base}}(G_2)) \subseteq L(G_2)$ .

Consider the context-free grammar  $\underline{\text{base}}(G_2)$ . It is well known that one can adopt for context-free grammars any strategy to select in a sentential form a non-terminal to rewrite. As long as this strategy does not lead to failures, the language will not be changed. From Lemma 4.6 it follows that in  $\underline{\text{base}}(G_2)$  the central derivation strategy of picking out non-terminals does not fail and if  $P_2 = \emptyset$ , then  $L(G_2) = \emptyset = L(\underline{\text{base}}(G_2))$ .

Hence  $L(\underline{\text{base}}(G_2)) \subseteq L(G_2)$ . □

Theorem 4.2.  $L(G) \in L(\text{CF})$ .

Proof. By Theorem 4.1.,  $L(G) = L(G_2)$  and Lemma 4.7 implies that  $L(G_2) \in L(\text{CF})$ . Hence  $L(G) \in L(\text{CF})$ . □



## 5. THE MAIN RESULT AND CONCLUSIONS

Now we can prove that all 1S grammars generate context-free languages only.

Theorem 5.1.  $L(1S) = L(CF)$ .

Proof. That  $L(CF) \subseteq L(1S)$  is clear.

From Theorem 2.1 and Theorem 4.2 it follows that for every 1S grammar  $H$  there exists a CF grammar  $G_H$  and a homomorphism  $h$  such that  $L(H) = h(L(G_H))$ . Since  $L(CF)$  is closed under homomorphisms,  $L(H) \in L(CF)$ .

Thus,  $L(1S) \subseteq L(CF)$ . □

Corollary 5.1. Every EOS based 1S grammar generates a context-free language. □

It is important to note at this point that all our constructions (in particular in the proof of Theorem 2.1) have been effective. Given a homomorphism  $h$  and a context-free grammar  $H_1$  one can also effectively construct a context-free grammar  $H_2$  such that  $L(H_2) = h(L(H_1))$ . As a consequence we have the following.

Theorem 5.2. Given a (EOS based) 1S grammar  $G$  it is decidable, whether or not

(1).  $L(G)$  is empty,

(2).  $L(G)$  is finite. □

Another consequence is the following.

Theorem 5.3. For every (EOS based) 1S grammar there exists an equivalent (EOS based) 1S grammar without erasing productions ( $\Delta \notin \text{rhs}(P)$ , where  $P$  is its set of productions). □

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ADDING GLOBAL FORBIDDING  
CONTEXT TO CONTEXT-FREE GRAMMARS

by

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