A NUMBER SEQUENCE RELATING TO THE CLOSEPACKING OF PRIMES

by

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Abstract.

This paper introduces the sequence of integers, 0, 2, 6, 8, 12, 18, ... with the properties that (1) it is "evasive", i.e., it does not form a complete system of residues modulo any prime, and (2) it is as densely packed as possible, subject to the constraint of evasiveness. The terms of the sequence specify how closely packed the primes can be in the "sufficiently large" case; that is (by conjecture), it is possible to have arbitrarily large primes of the form p, p+2, p+6, ... but no closer packing such as p, p+1 is allowed, except in finitely many cases at most. The order of a prime p is defined as the number of successive terms of the closepacking sequence that are reproduced by subtracting p from itself and its successors; a similar definition of order is applied to the terms of the sequence themselves. It is shown that, except for the first closepacking number, 0, no term of the sequence nor any prime has infinite order. However it is conjectured that there are primes and closepacking numbers of all finite orders, and moreover, that the frequencies of occurrence of corresponding orders in the two sequences are asymptotically equal. These conjectures are supported by computational evidence.

The famous unproven twin prime conjecture asserts that there are infinitely many prime number pairs of the form p, p+2. More generally there seems to be an infinity of prime sequences of the form $p+x_0$, $p+x_1$, ..., $p+x_n$ for any finite set of integers which do not form a complete system of residues modulo any prime. Thus there appear to be infinitely many prime triples of the form p, p+4, p+6, or quadruples p, p+2, p+6, p+8. On the other hand, if the x_i do form a complete system for some prime q then q must divide some term of any associated prime sequence so that (1) this term must equal q, consequently, (2) only finitely many such sequences can exist. (But such a sequence need not be unique, as was pointed out to me by Wolfgang Schmidt. For example the prime sequences 3, 5, 11, 17, 29 and 5, 7, 13, 19, 31 both form complete systems modulo 5.)

A set S of integers which does not form a complete system of residues modulo an integer q is said to evade q. S will be called evasive if it evades all $q > 1$, and we then write Ev(S). Thus the above conjectures assert that there are infinitely many prime sequences for any finite evasive set S. (If S is infinite, however, there need not be any associated prime sequence, as is shown later.)

Clearly a set is evasive iff it evades every prime. Moreover, a finite set S is evasive iff it evades every prime $\leq$ the number of elements in S, $|S|$. Thus there is an effective test for evasiveness of finite sets. A finite evasive set can always be enlarged, as is shown by considering values $x = y+kt$ where y is any element of the given set S, k is any integer, and t is the product of all primes $\leq |S|+1$. Then for k sufficiently large, $x \notin S$ but x will have the same residues as y modulo any prime $\leq |S \cup \{x\}|$, hence $Ev(S \cup \{x\})$.

Given any evasive set S, an integer $x \notin S$ is assimilable if $S \cup \{x\}$ is also evasive. S is said to be closepacked if there is no assimilable x in the smallest interval containing S. Thus a finite evasive set S is closepacked iff there is no assimilable $x \in [\min(S), \max(S)]$. Examples of closepacked evasive sets which were considered implicitly in the prime conjectures above are $\{0, 2\}$, $\{0, 4, 6\}$, and $\{0, 2, 6, 8\}$. For such sets the associated sequences for all sufficiently large primes are also closepacked in the sense that no additional prime can be inserted between the smallest and largest terms. As in the general case of an evasive set a finite closepacked set can always be enlarged by adding the smallest assimilable element larger than the maximum of the set (or the largest assimilable element smaller than the minimum).

Here we shall consider the sequence of integers c_0, c_1, ..., obtained by starting with $c_0 = 0$ and, for $n > 0$, defining c_n as the smallest positive, assimilable value for the set $\{c_0, c_1, \ldots, c_{n-1}\}$. Then (since $\{0\}$ is evasive) the c_i form an infinite evasive closepacked set, with the additional property that any finite initial segment $\{c_0, c_1, \ldots, c_n\}$ is also evasive and closepacked. The sequence of values c_i will be called the closepacking sequence (CPS); each term is referred to as a closepacking number (cpn). The first few cpn's are 0, 2, 6, 8, 12, 18, 20, 26, 30, 32; a more extensive tabulation is shown in fig. 1. In particular, given that $c_0 = 0$, every cpn must be even to evade 2. Thus 1 is the (unique) "forbidden residue" of the CPS modulo 2. Other primes too must all have forbidden residues to preserve evasiveness. For example we find that 3, 5, 7, and 11 have the unique forbidden residues 1, 4, 3, and 5, respectively. It appears that every prime must have a unique forbidden residue though this has not been proved.

However this conjecture can be readily verified for successive primes by computation. In doing so we also compute the "exhaustion number" or number of terms of the sequence, taken in order, that are needed to exhaust all but one of the residues so that the one remaining is uniquely forbidden. For example, the exhaustion numbers for 2, 3, 5, 7, and 11 are, respectively, 1, 2, 4, 6, and 11. A more extensive tabulation is shown in figure 2.

In particular it appears that the exhaustion number for the nth prime p_n is about what would be expected assuming that the residues for the CPS are randomly distributed for the given prime. That is, the expected number of cpn's needed to exhaust all the p_n-1 available residues is about $p_n \ln(p_n)$. This is roughly confirmed in the tabulation, though the exhaustion numbers appear to be somewhat smaller, particularly near the beginning of the sequence. This could be accounted for by the high density of cpn's near the beginning of the sequence and the fact that all cpn's are even, so that for an odd prime p_n, all cpn's $< 3p_n$ will have distinct residues, leading to a more rapid exhaustion than would follow from random selection.

On the other hand, the forbidden residues might be expected to be randomly distributed, particularly since the cpn's are expected to have the same asymptotic density distribution as the prime numbers (cf. later discussion). In fact the tabulation shows a preponderance of odd over even residues near the beginning, but this seems to be slowly leveling out. The scarcity of even forbidden residues can be accounted for, again, by the high density of cpn's near the beginning of the sequence and the fact that all cpn's are even. Thus, for small primes at least, many more even than odd residues are initially assigned by the CPS, so that the last remaining residue is more likely to be odd. On the other hand, for a large prime p_n, the number of the initial cpn's having even residues, that is, those $< p_n$, is expected to be only about $p_n/\ln(p_n)$, which is small enough that its effect on the forbidden residue should asymptotically vanish.

An interesting concept which derives from the CPS is that of the <u>order</u> of a prime number p_n, defined as the largest m such that $p_{n+k} = p_n + c_k$ whenever $0 \leq k < m$. We then write $o(p_n) = m$. In some sense the order furnishes a measure of how closely packed the primes are starting at p_n, at least in the "sufficiently large" case, since here no closer packing is possible. Anomalous closepacking occurs near the beginning of the prime sequence, however, as is illustrated by the examples of 2 and 3, or 3, 5, and 7.

Thus, given prime p_n with $o(p_n) = m$, the fact that $p_{n+m} \neq p_n + c_m$ (which must follow if the order is m) can occur either because p_{n+m} is too large (the usual case) or too small (the anomalous case). The only anomalous primes in this sense appear to be 2, 3, 5, and 11 though, like many other conjectures, this remains unproven. In particular 11 has the spectacularly high order of 15, as can be verified from the figures; possibly this is the highest for any known prime. This fact allows an easy computation of the cpn's up to $c_{14} = 56$, using the relation $c_n = p_{n+5} - 11$. (A "weak" order can also be defined as the largest m such that $p_n + c_k$ is prime (but not necessarily $= p_{n+k}$) whenever $0 \leq k < m$. For this case the order of 11 is an even larger 24; nonanomalous primes will have the same order as before, however.)

Based on the prime conjectures at the beginning, we expect that there are infinitely many primes of all finite orders. (A prime can be found with order exactly m by associating a prime sequence for an evasive set containing all the cpn's up to c_{m-1}, but for which c_m is omitted and moreover, is not assimilable. Such a set is constructible by an extension of the method noted earlier for enlarging a finite evasive set.) It can be shown, however, that <u>there is no prime of infinite order.</u>

This follows from the well-known arithmetic progression theorem which, in the form useful here, asserts that the primes greater than or equal to a given prime p_n must form a complete system of residues modulo p_n. (The residue 0 will occur only once, namely, for p_n itself, while all other residues will occur infinitely often, with asymptotically equal frequencies.) If p_n had infinite order on the other hand, then we would have $p_{n+k} = p_n + c_k$ for all $k \geq 0$, so that the residues would be identical to those for the cpn's. Since the latter form an evasive set however, not all residues could be present, contradicting the arithmetic progression theorem. In particular the cpn's furnish an example of an infinite evasive set with no associated prime sequence.

On inspection we note that high-order primes are rare. Those of order ≥ 4 up to 10000 (with the order in parentheses) are 5 (5), 11 (15), 101 (5), 191 (4), 821 (4), 1481 (6), 1871 (4), 2081 (4), 3251 (4), 3461 (4), 5651 (4), 9431 (4). (These primes are easy to spot in a table because -- except for 5 -- they each begin a sequence of primes whose last digits are 1, 3, 7, 9.) Some larger primes which would have order ≥ 8 are noted in [1].

As in the case of the primes we can define the order of the cpn c_n by $o(c_n) = m$ where m is as large as possible such that $c_n + c_k = c_{n+k}$ whenever $0 \leq k < m$. (Fortunately under this definition $o(2) = 1$ whether 2 is regarded as a prime or a cpn; otherwise no prime can be a cpn, thus order is well-defined.) Thus for a prime p_n of high order, the order of the immediately following primes p_{n+1}, p_{n+2}, ... must be the same as for the cpn's c_1, c_2, ...

It will be noted that cpn 0 must have infinite order in contrast to the primes which, as we have shown, always have finite order. It is easily shown, however, that no other cpn has infinite order. If there existed another cpn c_n of infinite order then it would follow that $c_{mn} = mc_n$ for all $m > 0$, i.e., all positive multiples of c_n would occur among the cpn's. But since c_n itself must be positive (since $c_n \neq 0$) this would contradict the evasiveness of the cpn's.

An interesting property of the cpn's is that there can be no anomalous closepacking as occurs with small primes, this being a direct consequence of evasiveness. This seems to preclude the occurrence of high-order cpn's near the beginning of the sequence (except for 0), but the occurrence of the higher orders more nearly approaches that of the primes if a larger segment of the CPS is considered. The cpn's of order ≥ 4 up to 10000 (again with the order in parentheses) are 0 (∞), 420 (4), 1980 (4), 2070 (4), 3780 (5), 5850 (4), 6810 (4), 9120 (5). (As in the case of the primes these are easy to spot in a table, the digit sequence in this case being 0, 2, 6, 8.) Thus it seems reasonable to conjecture that, as with the primes, there are infinitely many cpn's of all finite orders. Continuing in a similar vein we can speculate that (1) the density of cpn's as subset of the integers approaches that of the primes (i.e., the ratio of densities approaches 1), and similarly (2) the density of cpn's of a given, finite order approaches that of the primes of the same order. These latter conjectures are suggested by the similarity of the rule for generating the cpn's (essentially a sieve) to that for the primes.

To test these conjectures (and to obtain other results for the cpn's) a computer program was written to compute the cpn's up to 10000 and higher. Counts of primes and cpn's of different orders are shown in fig. 3. Generally there is a close correlation between the corresponding groups for the two sequences, though the cpn's persistently show a smaller number of examples in each classification (at least when the order is not too large). Perhaps this is accounted for by the observation that, inasmuch as anomalous closepacking is allowed in the primes but not in the cpn's, it is also reasonable to find more examples of legitimate high-order closepacking among the primes. But the tabulations do support the conjecture that primes and cpn's of a given order have asymptotically equal frequencies.

At any rate, the cpn's appear to offer an interesting field of inquiry both for the experimentalist and theoretician.

	0	1	2	3	4	5	6	7	8	9
0	0	2	6	8	12	18	20	26	30	32
1	36	42	48	50	56	62	68	72	78	86
2	90	96	98	102	110	116	120	128	132	138
3	140	146	152	156	158	162	168	176	182	186
4	188	198	200	210	212	216	230	240	242	246
5	252	260	266	270	272	278	282	288	306	308
6	312	320	336	338	342	348	350	362	372	380
7	386	392	396	410	420	422	426	428	438	450
8	452	462	468	470	476	488	492	498	506	510
9	512	516	530	536	548	552	558	572	578	582
10	590	596	600	606	608	618	620	630	642	648
11	650	656	660	672	680	686	702	708	722	726
12	732	740	746	756	758	762	776	782	798	800
13	812	818	828	842	848	858	860	870	876	882
14	888	890	900	912	926	930	936	938	960	966
15	968	972	980	986	992	996	998	1008	1020	1022
16	1026	1052	1056	1058	1062	1068	1070	1076	1082	1086
17	1110	1118	1122	1128	1136	1140	1142	1148	1152	1166
18	1170	1176	1178	1188	1190	1196	1208	1212	1218	1220
19	1232	1238	1260	1266	1278	1280	1290	1296	1302	1308
20	1310	1332	1338	1346	1352	1362	1370	1376	1398	1400
21	1406	1412	1416	1428	1430	1436	1442	1458	1478	1488
22	1500	1502	1518	1532	1538	1542	1560	1566	1572	1580
23	1590	1602	1608	1610	1616	1628	1632	1640	1646	1650
24	1670	1680	1682	1686	1706	1712	1716	1722	1728	1742
25	1748	1752	1766	1770	1778	1790	1796	1800	1806	1812
26	1818	1832	1836	1848	1850	1862	1866	1868	1878	1880
27	1892	1896	1898	1902	1916	1922	1926	1946	1958	1980
28	1982	1986	1988	2000	2010	2028	2036	2046	2052	2060
29	2066	2070	2072	2076	2078	2088	2090	2100	2102	2108
30	2112	2130	2136	2148	2160	2162	2168	2178	2190	2192
31	2202	2210	2220	2226	2228	2240	2246	2252	2256	2262
32	2280	2286	2312	2318	2322	2342	2352	2358	2360	2366
33	2382	2388	2396	2400	2408	2412	2430	2442	2450	2456
34	2462	2466	2468	2478	2486	2496	2508	2528	2532	2552
35	2556	2570	2576	2580	2582	2592	2598	2616	2618	2622
36	2636	2648	2658	2672	2682	2690	2696	2706	2708	2720
37	2732	2738	2742	2750	2756	2762	2772	2786	2790	2792
38	2798	2820	2826	2828	2850	2856	2862	2868	2870	2888
39	2892	2910	2912	2918	2928	2930	2940	2958	2966	2970
40	2972	2976	2990	2996	3002	3026	3032	3036	3042	3060
41	3066	3072	3086	3092	3098	3110	3122	3126	3138	3150
42	3168	3176	3180	3192	3198	3200	3210	3222	3236	3240
43	3242	3260	3276	3282	3308	3312	3318	3330	3336	3348
44	3350	3362	3366	3378	3380	3386	3396	3402	3408	3422
45	3428	3432	3452	3462	3480	3488	3500	3506	3512	3516
46	3522	3528	3540	3542	3570	3578	3588	3596	3606	3620
47	3626	3630	3632	3648	3660	3666	3672	3686	3698	3702
48	3710	3716	3722	3726	3728	3738	3740	3768	3780	3782
49	3786	3788	3792	3812	3816	3840	3842	3858	3870	3876
50	3878	3896	3906	3912	3920	3926	3936	3938	3948	3950
51	3960	3962	3978	3996	4008	4010	4022	4026	4032	4038
52	4040	4046	4052	4058	4062	4080	4088	4092	4110	4116
53	4118	4136	4146	4158	4160	4166	4176	4178	4188	4190
54	4202	4212	4232	4242	4248	4250	4256	4260	4268	4272
55	4286	4292	4296	4298	4302	4310	4316	4320	4326	4332
56	4338	4352	4368	4370	4380	4412	4418	4422	4428	4442
57	4446	4452	4458	4466	4470	4472	4500	4502	4506	4530
58	4556	4562	4568	4586	4596	4598	4608	4620	4628	4632
59	4638	4640	4652	4670	4692	4698	4706	4712	4718	4722

Fig. 1a. Closepacking numbers c_n for n = 0 to 599.

	0	1	2	3	4	5	6	7	8	9
60	4730	4736	4766	4776	4782	4788	4796	4800	4802	4830
61	4836	4842	4848	4860	4862	4866	4872	4890	4898	4908
62	4916	4940	4946	4950	4970	4976	4982	4986	4992	4998
63	5012	5018	5030	5040	5046	5060	5070	5088	5096	5108
64	5118	5126	5132	5150	5156	5160	5166	5168	5178	5198
65	5202	5210	5222	5228	5238	5256	5258	5262	5268	5276
66	5282	5286	5300	5306	5312	5322	5336	5342	5352	5376
67	5382	5388	5402	5408	5412	5418	5426	5432	5438	5468
68	5478	5480	5486	5492	5496	5520	5528	5550	5556	5558
69	5562	5570	5580	5586	5588	5592	5598	5606	5612	5618
70	5628	5636	5640	5642	5658	5672	5676	5678	5696	5700
71	5706	5718	5720	5726	5730	5732	5742	5748	5756	5762
72	5772	5790	5798	5810	5816	5822	5828	5832	5840	5850
73	5852	5856	5858	5882	5886	5888	5892	5900	5910	5952
74	5966	5976	5982	6006	6026	6036	6038	6048	6050	6056
75	6062	6068	6078	6080	6090	6092	6096	6102	6116	6120
76	6140	6162	6168	6182	6186	6188	6192	6200	6206	6210
77	6216	6218	6230	6236	6248	6252	6258	6272	6276	6278
78	6300	6302	6312	6320	6326	6350	6356	6362	6372	6378
79	6390	6392	6420	6426	6446	6452	6456	6458	6468	6476
80	6480	6486	6498	6500	6512	6516	6522	6530	6536	6542
81	6546	6558	6560	6570	6588	6606	6626	6630	6636	6642
82	6662	6668	6672	6690	6692	6698	6708	6710	6720	6728
83	6732	6738	6740	6752	6768	6776	6788	6798	6806	6810
84	6812	6816	6818	6846	6848	6876	6890	6896	6900	6908
85	6918	6920	6936	6938	6948	6950	6962	6966	6972	6978
86	6986	6992	6998	7002	7016	7022	7040	7046	7076	7082
87	7088	7098	7110	7140	7146	7152	7160	7170	7182	7196
88	7200	7208	7212	7218	7226	7230	7236	7260	7272	7278
89	7280	7296	7310	7316	7326	7328	7338	7340	7350	7358
90	7370	7376	7380	7382	7392	7406	7410	7412	7418	7422
91	7442	7446	7448	7466	7482	7488	7506	7508	7512	7520
92	7526	7530	7550	7560	7566	7568	7586	7592	7596	7602
93	7608	7620	7622	7638	7646	7656	7662	7686	7688	7698
94	7706	7722	7728	7730	7740	7746	7748	7770	7772	7800
95	7802	7806	7812	7820	7830	7832	7838	7842	7862	7880
96	7886	7896	7898	7902	7908	7910	7922	7938	7950	7952
97	7956	7986	7998	8010	8012	8028	8030	8036	8040	8042
98	8066	8072	8076	8096	8108	8118	8120	8132	8142	8162
99	8166	8180	8192	8202	8208	8216	8220	8238	8240	8250
100	8276	8280	8300	8306	8316	8322	8328	8346	8366	8370
101	8372	8378	8390	8402	8460	8462	8468	8472	8496	8510
102	8520	8526	8528	8532	8538	8546	8556	8562	8570	8588
103	8610	8622	8628	8630	8642	8652	8658	8666	8678	8688
104	8700	8708	8736	8738	8742	8748	8766	8778	8786	8790
105	8792	8798	8822	8832	8840	8846	8852	8856	8862	8888
106	8892	8916	8918	8930	8936	8940	8946	8952	8960	8976
107	8978	8982	8990	9006	9018	9020	9030	9038	9042	9048
108	9060	9066	9086	9090	9092	9116	9120	9122	9126	9128
109	9132	9150	9158	9162	9176	9198	9206	9210	9216	9230
110	9240	9242	9248	9252	9258	9260	9288	9290	9308	9312
111	9318	9330	9332	9350	9356	9360	9368	9378	9380	9396
112	9398	9422	9428	9456	9458	9462	9482	9486	9500	9522
113	9536	9546	9548	9552	9560	9570	9590	9596	9606	9618
114	9626	9632	9636	9638	9650	9662	9666	9668	9672	9678
115	9692	9702	9708	9710	9716	9720	9722	9728	9746	9750
116	9776	9788	9792	9818	9822	9830	9840	9858	9876	9888
117	9900	9902	9906	9926	9930	9932	9948	9956	9962	9968
118	9980	9990	9996	10008	10010	10032	10038	10052	10068	10080
119	10082	10100	10106	10110	10112	10116	10142	10148	10166	10172

Fig 1b. Closepacking numbers c_n for n = 600 to 1199.

n	p_n	f_n	e_n	n	p_n	f_n	e_n
1	2	1	1	26	101	13	225
2	3	1	2	27	103	51	312
3	5	4	4	28	107	53	337
4	7	3	6	29	109	57	234
5	11	5	11	30	113	29	293
6	13	1	14	31	127	63	462
7	17	7	19	32	131	65	471
8	19	9	37	33	137	43	434
9	23	11	38	34	139	69	535
10	29	25	53	35	149	119	349
11	31	15	50	36	151	75	458
12	37	33	57	37	157	122	470
13	41	13	80	38	163	81	489
14	43	21	81	39	167	83	477
15	47	23	99	40	173	112	413
16	53	31	125	41	179	89	527
17	59	29	131	42	181	4	474
18	61	52	213	43	191	95	619
19	67	33	156	44	193	94	539
20	71	35	330	45	197	174	554
21	73	35	161	46	199	99	666
22	79	39	220	47	211	105	743
23	83	41	173	48	223	111	690
24	89	58	207	49	227	113	1295
25	97	11	244	50	229	123	740

Fig. 2. Primes (p_n), forbidden residues (f_n) and exhaustion numbers (e_n), for $n = 1$ to 50.

n	order ≥1	≥2	≥3	≥4	≥5	≥6	≥7
1000	168	35	15	5	3	1	1
2000	303	61	24	7	4	2	1
3000	430	82	29	8	4	2	1
4000	550	103	34	10	4	2	1
5000	669	126	41	10	4	2	1
6000	783	143	45	11	4	2	1
7000	900	162	47	11	4	2	1
8000	1007	175	48	11	4	2	1
9000	1117	190	53	11	4	2	1
10000	1229	205	55	12	4	2	1

(a) primes

n	order ≥1	≥2	≥3	≥4	≥5	≥6	≥7
1000	157	31	13	2	1	1	1
2000	284	51	18	3	1	1	1
3000	404	69	21	4	1	1	1
4000	514	84	23	5	2	1	1
5000	630	99	26	5	2	1	1
6000	743	111	31	6	2	1	1
7000	863	132	34	7	2	1	1
8000	973	150	38	7	2	1	1
9000	1073	163	41	7	2	1	1
10000	1183	181	47	8	3	1	1

(b) cpn's.

fig. 3. Number of primes and cpn's up to n, for order as indicated.

References

[1] Gardner, M. Mathematical Games. Scientific American 244 No. 4 (1981) p. 26.

[2] Table of primes in "Handbook of Mathematical Functions" (M. Abramowitz and I. A. Stegun, eds.) National Bureau of Standards Applied Mathematics Series 55 (1964) p. 870.

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