ON FINITE SETS TESTING SQUARE FREE PROPERTY FOR ALL HOMOMORPHIMSMS BETWEEN TWO GIVEN ALPHABETS

by

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ABSTRACT

A nonempty word w is called ***square free*** if it cannot be written in the form w_1xxw_2 for words w_1, x, w_2 where x is nonempty; the set of all square free words over an alphabet Σ is denoted by $SF(\Sigma^+)$. A homomorphism $h:\Sigma^+ \to \Delta^+$ is called ***square free*** if $h(SF(\Sigma^+)) \subseteq SF(\Delta^+)$. Let Σ, Δ be finite alphabets. Then a set $X \subseteq SF(\Sigma^+)$ is called a (Σ, Δ)*–test set (of square freeness)* if for each homomorphism $g:\Sigma^+ \to \Delta^+$ the following holds: g is square free if and only if $g(X) \subseteq SF(\Delta^+)$; the family of all (Σ, Δ)*–test sets* is denoted by $TEST(\Sigma, \Delta)$. We demonstrate that $TEST(\Sigma, \Delta)$ contains a finite set if and only if either the cardinality of Σ is not bigger than 3 or the cardinality of Δ is not bigger than 2.

INTRODUCTION

The topic of repetitions of subwords in words initiated by A. Thue in [*T*] has turned out to be of interest in several areas of mathematics and in formal language theory (see, e.g., [*BEM*], [*C*], [*D*], [*MH*], and [*S1*]. The paper [*B*] by J. Berstel has pointed out quite deep connections between "Thue problems" and modern formal language theory; since then this problem area became quite active within formal language theory (see, e.g., [*Br*], [*Cr*], [*K*] and [*S2*]). In particular the topic of square free homomorphisms received a lot of new attention.

A nonempty word w is called ***square free*** if it cannot be written in the form w_1xxw_2 for words w_1, x, w_2 where x is nonempty; the set of all square free words over an alphabet Σ is denoted by $SF(\Sigma^+)$. A homomorphism $h:\Sigma^+ \to \Delta^+$ is called ***square free*** if $h(SF(\Sigma^+)) \subseteq SF(\Delta^+)$. A number of recent papers (see, e.g., [*B*], [*Cr*], [*K*] and [*ER*]) is concerned with the problem of testing the square free property of a homomorphism.

Informally speaking a set $X \subseteq SF(\Sigma^+)$ is a ***(square freeness) test set for a homomorphism*** $h:\Sigma^+ \to \Delta^+$ if : h is square free if and only if $h(X) \subseteq SF(\Delta^+)$. We will say that a test set X for h is ***h–independent*** if X is a subset of $SF(\Sigma^+)$ defined using the cardinalities of Σ and Δ only; otherwise X is ***h-dependent***. In the literature both homomorphism dependent and homomorphism independent test sets are investigated.

In particular it seems natural to ask about the existence of *finite* test sets which are homomorphism independent. This issue is settled in our paper. We prove that (for all homomorphisms from Σ^* into Δ^*) homomorphism independent finite test sets exist if and only if either the cardinality of Σ is not bigger than 3 or the cardinality of Δ is not bigger than 2.

PRELIMINARIES

We will use mostly standard language theoretic notation and terminology. For a finite set A, $\#A$ denotes its cardinality and for sets A, B, $A-B$ denotes the set theoretic difference of A and B.

For a word x: $|x|$ denotes its length, $first(x)$ denotes the first letter of x, $last(x)$ denotes the last letter of x, $alph(x)$ denotes the set of all letter occurring in x and, for a letter b, $\#_b(x)$ denotes the number of occurrences of b in x. Λ denotes the empty word. For an alphabet Σ, Σ^+ denotes the set of all nonempty words over Σ and $\Sigma^* = \Sigma^+ \cup \{\Lambda\}$. Given words x and y, we say that x is a *subword* of y, written x *sub* y, if $y = y_1 x y_2$ for some words y_1, y_2. (Subwords are sometimes referred to also as *segments* or *factors*).

For alphabets Σ, Δ, $HOM(\Sigma^+, \Delta^+)$ denotes the family of all homomorphisms from Σ^+ into Δ^+.

A nonempty word y is called a *square* if $y = y_1 x x y_2$ for some words y_1, y_2, x where $x \neq \Lambda$, otherwise y is called *square free;* the set of all square free words over Σ is denoted by $SF(\Sigma^+)$.

A homomorphism $h \in HOM(\Sigma^+, \Delta^+)$ is called *square free* if $h(SF(\Sigma^+)) \subseteq SF(\Delta^+)$.

For the considerations of this paper it is convenient to adopt the following convention.

Let $\Sigma_\omega = \{a_1, a_2, \cdots\}$ be a fixed (ordered) countable alphabet. Then, for each $n \geq 1$, let $\Sigma_n = \{a_1, \ldots, a_n\}$.

For $n, l \geq 1$, $T_{n,l} = \{w \in SF(\Sigma_n^+) : |w| \leq l\}$ and, for a homomorphism h of Σ_n^+,

$$T_h = \{w \in SF(\Sigma_n^+): \text{there exist } a, b \in \Sigma \text{ and } u \in \Sigma_n^* \text{ such that } w = a\,u\,b \text{ and either } h(u) \textit{ sub } h(a) \text{ or } h(u) \textit{ sub } h(b)\} \cup \Sigma_n.$$

The following result was proved in [*ER*].

Proposition 1. Let $h \in HOM(\Sigma_n^+, \Sigma_m^+)$ for some $n, m \geq 1$. Then h is square free if and only if $h(T_{n,3} \cup T_h) \subseteq SF(\Sigma_m^+)$. ■

Also the following result proved in [*BEM*] will be useful in the sequel.

Proposition 2. Let $n, m \geq 3$. Then there exists an $h \in HOM(\Sigma_n^+, \Sigma_m^+)$ such that h is square free. ∎

In this paper we will be concerned with the problem of testing the square freeness of a homomorphism. In particular we will consider the problem of the existence of *finite* test sets which for given alphabets Σ_n and Σ_m would "verify" whether or not an arbitrary $h \in HOM(\Sigma_n^+, \Sigma_m^+)$ is square free. The family of such test sets is formally defined as follows.

Let $n, m \geq 1$. A set $X \subseteq SF(\Sigma_n^+)$ is a (n, m) *test set (of square freeness)* if for each homomorphism $h \in HOM(\Sigma_n^+, \Sigma_m^+)$ the following holds:
h is square free if and only if $h(X) \subseteq SF(\Sigma_m^+)$.

We would like to conclude this section by the following remark.

In order to simplify the notation and avoid very cumbersome formulations we will often not distinguish between subwords and their occurrences in words (this is quite customary in formal language theory). This should not lead to a confusion because the exact meaning should be always clear from the context. Moreover to avoid misunderstanding we often provide figures that illustrate the situations considered.

THE THEOREM

In this section we provide necessary and sufficient conditions for $TEST(n, m)$ to contain finite sets. Those conditions are given by the following result.

Theorem. For each $n, m \in \mathbf{N}^+$, $TEST(n, m)$ contains a finite set if and only if either $n \leq 3$ or $m \leq 2$.

Proof:

Since it is obvious that if either $n = 1$ or $m = 1$ then $TEST(n, m)$ contains a finite set, throughout the proof of this theorem we will assume that $n, m \geq 2$.

First we prove the "if" part of the theorem.

Lemma 1. If $n \leq 2$ then, for each $m \geq 2$, $TEST(n, m)$ contains a finite set.

Proof of Lemma 1:

If $n \leq 2$ then $SF(\Sigma_n^+)$ is a finite set. Since $SF(\Sigma_n^+) \in TEST(n, m)$, Lemma 1 holds. ■

Lemma 2. If $n = 3$ then, for each $m \geq 2$, $TEST(n, m)$ contains a finite set.

Proof of Lemma 2:

Consider $T_{3,5}$.

Let $h \in HOM(\Sigma_3^+,\Sigma_m^+)$ where $m \geq 2$. Let $w \in T_{3,3} \cup T_h$.

If $|w| \leq 3$ then $w \in T_{3,5}$.....................................(1)

Assume then that $|w| > 3$. Hence by the definition of T_h either $w = a\,u\,b$ or $w = b\,u\,a$ for some $a, b \in \Sigma_3$ and $u \in \Sigma_3^+$ such that $h(u)\ sub\ h(a)$. If $a \in alph(u)$ then $h(u) = h(a)$ and consequently $a = u$ contradicting the fact that $w \in SF(\Sigma_3^+)$; thus it must be that $a \notin alph(u)$. Consequently $\#alph(u) \leq 2$ and, since $u \in SF(\Sigma_3^+)$, $|u| \leq 3$ and $|w| \leq 5$. Thus if $|w| > 3$ then $w \in T_{3,5}$..(2)

From (1) and (2) it follows that $T_{3,3} \cup T_h \subseteq T_{3,5}$ and consequently, by Proposition 1, $TEST(3, m)$ contains $T_{3,5}$ which is a finite set.

Thus Lemma 2 holds. ■

Lemma 3. If $n \geq 3$ and $m = 2$ then $TEST(n, m)$ contains a finite set.

Proof of Lemma 3:

Consider $X = \{a_1 a_2 a_3 a_1\}$ and an arbitrary homomorphism $h \in HOM(\Sigma_n^+, \Sigma_2^+)$. Since $|h(a_1 a_2 a_3 a_1)| \geq 4$, $h(a_1 a_2 a_3 a_1) \notin SF(\Sigma_2^+)$. But it is easily seen that no homomorphism in $HOM(\Sigma_n^+, \Sigma_2^+)$ is square free and consequently X is an element of $TEST(n, 2)$.

Thus Lemma 3 holds. ■

Now the "if" part of the theorem follows from Lemma 1, Lemma 2 and Lemma 3.

We turn now to the "only if" part of the theorem.

Lemma 4. If $n \geq 4$ and $m \geq 3$ then $TEST(n, m)$ does not contain a finite set.

Proof of Lemma 4:

This lemma is proved through a sequence of claims as follows.

Claim 4.1. Let $n \geq 4$, $m \geq 3$ and $k \geq 3$ and let $X \in TEST(n, m)$. If $HOM(\Sigma_k^+, \Sigma_m^+)$ contains a square free homomorphism, then $X \in TEST(n, k)$.

Proof of Claim 4.1:

Let g be a square free homomorphism in $HOM(\Sigma_k^+, \Sigma_m^+)$.

Assume to the contrary that

$X \notin TEST(n, k)$..(3)

That is there exists an $h \in HOM(\Sigma_n^+, \Sigma_k^+)$ such that $h(X) \subseteq SF(\Sigma_k^+)$ and $h(w) \not\subseteq SF(\Sigma_k^+)$ for some word $w \in SF(\Sigma_n^+)$. Consider such a word w and consider $g\ h(w)$. Since $h(w) \not\subseteq SF(\Sigma_k^+)$, $g\ h(w) \not\subseteq SF(\Sigma_m^+)$. Moreover $gh(X) \subseteq SF(\Sigma_m^+)$, because $h(X) \subseteq SF(\Sigma_k^+)$ and g is square free. Hence on the one hand gh is not square free while on the other hand $gh(X) \subseteq SF(\Sigma_m^+)$; this contradicts the fact that $X \in TEST(n, m)$.

Consequently (3) cannot hold and Claim 4.1 is proved. ■

Claim 4.2 Let $n \geq 4$, $m \geq 3$ and $k \geq 3$. Then $TEST(n, m) = TEST(n, k)$.

Proof of Claim 4.2:

This follows directly from Claim 4.1 and Proposition 2. ■

Claim 4.3. Let $n \geq 4$ and $l \geq 1$. Then there exists an $h \in HOM(\Sigma_n^+, \Sigma_{n+1}^+)$ such that

(i) h is not square free, and

(ii) $h(T_{n,l}) \subseteq SF(\Sigma_{n+1}^+)$.

Proof of Claim 4.3:

Let w_0 be a fixed word from $T_{n-1,l}$ (since $n \geq 4$ such a w_0 exists). Let h be the homomorphism of Σ_n defined as follows:

$h(a_i) = a_i$ for $1 \leq i \leq n-1$ and

$h(a_n) = a_n\ a_{n+1}\ w_0\ a_{n+1}$.

We will demonstrate now that h satisfies conditions (i) and (ii) of the statement of Claim 4.3.

ad(i). Consider the word $u = a_n w_0$. Since $a_n \not\in alph(w_0)$ and w_0 is square free, $u \in SF(\Sigma_n^+)$. However $h(u) = a_n a_{n+1} w_0 a_{n+1} w_0$ is a square; thus h is not square free and (i) holds.

ad(ii). Let $g \in HOM(\Sigma_n^+, \Sigma_{n+2}^+)$ be the homomorphism defined by

$g(a_i) = a_i$ for $1 \le i \le n-1$, and

$g(a_n) = a_n\, a_{n+1}\, w_0\, a_{n+2}$.

Claim 4.3.1. g is square free.

Proof of Claim 4.3.1:

Claim 4.3.1.1. Let $Y = SF(\Sigma_n^+) \cap (\Sigma_{n-1}^* a_n \Sigma_{n-1}^* \cup \Sigma_{n-1}^*)$. If $w \in Y$, then $g(w) \in SF(\Sigma_{n+2}^+)$.

Proof of Claim 4.3.1.1:

Obviously

if $w \in SF(\Sigma_{n-1}^+)$, then $g(w) = w \in SF(\Sigma_{n+2}^+)$......................................(4)

Let us assume then that $w \in SF(\Sigma_n^+) \cap \Sigma_{n-1}^* a_n \Sigma_{n-1}^*$, say $w = w_1 a_n w_2$ for some $w_1, w_2 \in \Sigma_{n-1}^*$. Then from the definition of g it follows that $\#_{a_n}(g(w)) = 1$, $\#_{a_{n+1}}(g(w)) = 1$ and $\#_{a_{n+2}}(g(w)) = 1$. Thus if $g(w) = x_1\, xx\, x_2$ for some $x_1, x_2 \in \Sigma_{n+2}^*$ and $x \in \Sigma_{n+2}^+$, then $a_n, a_{n+1}, a_{n+2} \not\in alph(x)$. But $g(w) = g(w_1)\, g(a_n)\, g(w_2) = w_1\, g(a_n) w_2$. Consequently $g(w) \in SF(\Sigma_{n+2}^+)$. Hence we get

if $w \in SF(\Sigma_n^+) \cap \Sigma_{n-1}^* a_n \Sigma_{n-1}^*$, then $g(w) \in SF(\Sigma_{n+2}^+)$......................................(5)

Now Claim 4.3.1.1. follows from (4) and (5). ■

Claim 4.3.1.2. If $w \in (T_{n,3} \cup T_g) - Y$, then $g(w) \in SF(\Sigma_{n+2}^+)$.

Proof of Claim 4.3.1.2:

Since $w \not\in Y$, $\#_{a_n}(w) \ge 2$ which implies (because $w \in SF(\Sigma_n^+)$) that $|w| \ge 3$. Thus we have two cases to consider.

Case 1. $w = a_n\, a_i\, a_n$ where $1 \le i \le n-1$ and

Case 2. $w = a_n\, u\, a_n$ where $u \in \Sigma_n^+$ and $g(u)\ sub\ g(a_n)$. (Note that in this case-see the reasoning following (1) - $a_n \not\in alph(u)$ and consequently it must be that $u \in \Sigma_{n-1}^{+}$).

We will consider separately each of these two cases.

Case 1. $w = a_n\, a_i\, a_n$ where $1 \le i \le n-1$. Then $g(w) = g(a_n)\, a_i\, g(a_n)$. Assume that $g(w)$ is a square, that is, for some $x \in \Sigma_{n+2}^{+}$, xx *sub* $g(w)$. It is easily seen that neither xx *sub* $g(a_n)a_i$ nor xx *sub* $a_i g(a_n)$. Hence $a_n \in alph(x)$ and $a_{n+2} \in alph(x)$ which easily leads to the conclusion that $x = g(a_n)$; a contradiction.

Thus in this case we have $g(w) \in SF(\Sigma_{n+2}^{+})$......................................(6)

Case 2. $w = a_n\, u\, a_n$ where $u \in \Sigma_{n-1}^{+}$.

Again a reasoning similar to the one above leads one to the conclusion that in this case $g(w) \in SF(\Sigma_{n+2}^{+})$......................................(7)

Now Claim 4.3.1.2 follows from (6) and (7). ∎

Then Claim 4.3.1 follows from Claim 4.3.1.1, Claim 4.3.1.2 and Proposition 1.

∎

Claim 4.3.2. Let $w \in SF(\Sigma_n^{+})$. If $h(w) \notin SF(\Sigma_{n+1}^{+})$, then $|w| > l$.

Proof of Claim 4.3.2:

Let us consider $h(w)$ and $g(w)$. Let $f \in HOM(\Sigma_{n+2}^{+}, \Sigma_{n+1}^{+})$ be defined as follows: $f(a_i) = a_i$ for $1 \le i \le n+1$ and $f(a_{n+2}) = a_{n+1}$. Clearly $fg = h$.

Since $|g(a_i)| = |h(a_i)|$ for each $1 \le i \le n$, there is one-to-one correspondence between all occurrences of letter in $g(w)$ and all occurrences of letters in $h(w)$; actually each (occurrence of a) letter in $g(w)$ is mapped by f in the corresponding (occurrence of a) letter in $h(w)$. Since $h(w)$ is a square, $h(w) = u_1\, x_1\, x_2\, u_2$ for some $u_1, u_2 \in \Sigma_{n+1}^{*}$ and $x_1 = x_2 = x \in \Sigma_{n+1}^{+}$ (we have written $x_1 x_2$ rather than $x\, x$ so that we can easier talk about the first given occurrence of x and the second given occurrence of x!). Let then v_1 be the (occurrence of the) subword in $g(w)$ corresponding to (the occurrence of x given by) x_1 and let v_2 be the (occurrence of the) subword in $g(w)$ corresponding to (the occurrence of x given by) x_2.

The situation can be illustrated as follows:

Figure 1

where $f(v_1) = x_1$ and $f(v_2) = x_2$.......................................(8)

Since by Claim 4.3.1 g is square-free, $g(w)$ is not a square and consequently (8) implies that

$a_{n+2} \in alph(v_1 v_2)$.......................................(9)

Also from (8) it follows that $|v_1| = |v_2|$. Hence we can pair together:

the first (occurrence of a) letter of v_1 with the first (occurrence of a) letter of v_2,

the second (occurrence of a) letter of v_1 with the second (occurrence of a) letter of v_2,

....

....

the $|v_1|$-th (occurrence of a) letter of v_1 with the $|v_2|$-th (occurrence of a) letter of v_2.

Let cor_1 be this set of pairs.

Similarly we can pair together:

the first (occurrence of a) letter of x_1 with the first (occurrence of a) letter of x_2,

the second (occurrence of a) letter of x_1 with the second (occurrence of a) letter of x_2,

....

....

the $|x|$-th (occurrence of a) letter of x_1 with the $|x|$-th (occurrence of a) letter of x_2.

Let cor_2 be this set of pairs.

From (9) it follows that

either $(a_{n+1}, a_{n+2}) \in cor_1$ or $(a_{n+2}, a_{n+1}) \in cor_1$..........................(10)

Claim 4.3.2.1. If $(a_{n+1}, a_{n+2}) \in cor_1$, then

$(first(v_1), first(v_2)) = (a_{n+1}, a_{n+2})$.

Proof of Claim 4.3.2.1:

Assume to the contrary that

$(first(v_1), first(v_2)) \neq (a_{n+1}, a_{n+2})$......................................(11)

Consider the pair (d_1, d_2) where d_1 is an occurrence in v_1 immediately to the left of a_{n+1} and d_2 is an occurrence in v_2 immediately to the left of a_{n+2}. From the definition of g it follows immediately that $d_1 = a_n$ and $d_2 = last(w_0)$. Then, by (8), cor_2 must contain the pair $(f(a_n), f(last(w_0))) = (a_n, last(w_0))$ where $last(w_0) \in \Sigma_{n-1}$; a contradiction (since $(a_n, last(w_0)) \in cor_2$, it must be that $a_n = last(w_0)$).

Consequently (11) cannot hold and Claim 4.3.2.1 is proved. ∎

Similarly one proves the following result.

Claim 4.3.2.2. If $(a_{n+2}, a_{n+1}) \in cor_1$, then

$(first(v_1), first(v_2)) = (a_{n+2}, a_{n+1})$. ∎

From (10), Claim 4.3.2.1 and Claim 4.3.2.2 it follows that we have two cases to consider:

Case 3. $(first(v_1), first(v_2)) = (a_{n+2}, a_{n+1})$, and

Case 4. $(first(v_1), first(v_2)) = (a_{n+1}, a_{n+2})$.

We will consider separately each of these cases.

Case 3. $(first(v_1), first(v_2)) = (a_{n+2}, a_{n+1})$.

The situation can be illustrated as follows.

Figure 2.

Since $(first(v_1), first(v_2)) = (a_{n+2}, a_{n+1})$, (8) implies that $(first(x_1), first(x_2)) = (a_{n+1}, a_{n+1})$. By the definition of g, $first(v_1)$ is contributed (via g) by an occurrence of a_n in w; the same occurrence of a_n must contribute (via h) $first(x_1)$ in $h(w)$. Also $first(v_2)$ must be contributed (via g) by (a different from the above) occurrence of a_n in w; by the definition of g this occurrence of a_n in w will contribute (via g) immediately to the left of $first(v_2)$ an occurrence of a_n. Thus $last(v_1) = a_n$ and so, by (8), $last(x_1) = a_n$. Clearly the same occurrence of a_n in w contributes (via h) $last(x_1)$ and $first(x_2)$. Thus from the definition of h it follows that immediately to the right of $first(x_2)$ we have an occurrence of w_0; since $x_1 = x_2 = x$ it must be that immediately to the right of $first(x_1)$ there is an occurrence of w_0. Thus, by the definition of f, immediately to the right of $first(v_1)$ there is an occurrence of w_0.

Consequently $|w| > |a_n w_0| = 1 + l > l$ and so

Claim 4.3.2 holds in Case 3.......................................(12)

Case 4. $(first(v_1), first(v_2)) = (a_{n+1}, a_{n+2})$.

We will consider separately two subcases.

Case 4.1. Both $first(v_1)$ and $first(v_2)$ are contributed (via g) by the same occurrence of a_n in w.

Then the situation can be illustrated as follows:

Figure 3.

From the definition of g it follows that $v_1 = a_{n+1}\, w_0$. Thus from the definition of f it follows that $x_1 = a_{n+1}\, w_0$. Consequently $x_2 = a_{n+1} w_0$ and so from the definition of h it follows that in w immediately to the right of the given occurrence of a_n there is an occurrence of w_0.

Hence $|w| \geq |a_n w_0| = 1 + l > l$ and so

Claim 4.3.2 holds in Case 4.1..(13)

Case 4.2. $first(v_1)$ and $first(v_2)$ are contributed (via g) by different occurrences of a_n in w.

Then reasoning ambiguously to Case 3 we prove that

Claim 4.3.2 holds in Case 4.2.......................................(14)

Now Claim 4.3.2 follows from (12), (13) and (14). ■

Claim 4.3.2 implies that the property (ii) of the statement of Claim 4.3 holds.

Since we have also proved that the property (i) of this statement holds, Claim 4.3 holds. ■

Now we complete the proof of Lemma 4 as follows.

Let $n \geq 4$ and let $X \in TEST(n, n+1)$. If X is finite then, for some $l \geq 1$, $|x| \leq l$ for each $x \in X$. Then, by Claim 4.3, there exists a $h \in HOM(\Sigma_n^+, \Sigma_{n+1}^+)$ such that h is not square free but $h(x)$ is square free for each $x \in X$. Consequently we get a contradiction to the assumption that $X \in TEST(n,n+1)$. Thus X must be finite. Hence we have:

for each $n \geq 4$, if $X \in TEST(n,n+1)$ then X is infinite.............(15)

On the other hand Claim 4.2 implies that for each $n \geq 4$, $m \geq 3$,

$TEST(n, n+1) = TEST(n, m)$.......................................(16)

Lemma 4 follows from (15) and (16). ■

Since Lemma 4 implies the "only if" part of the theorem, the theorem holds.

■

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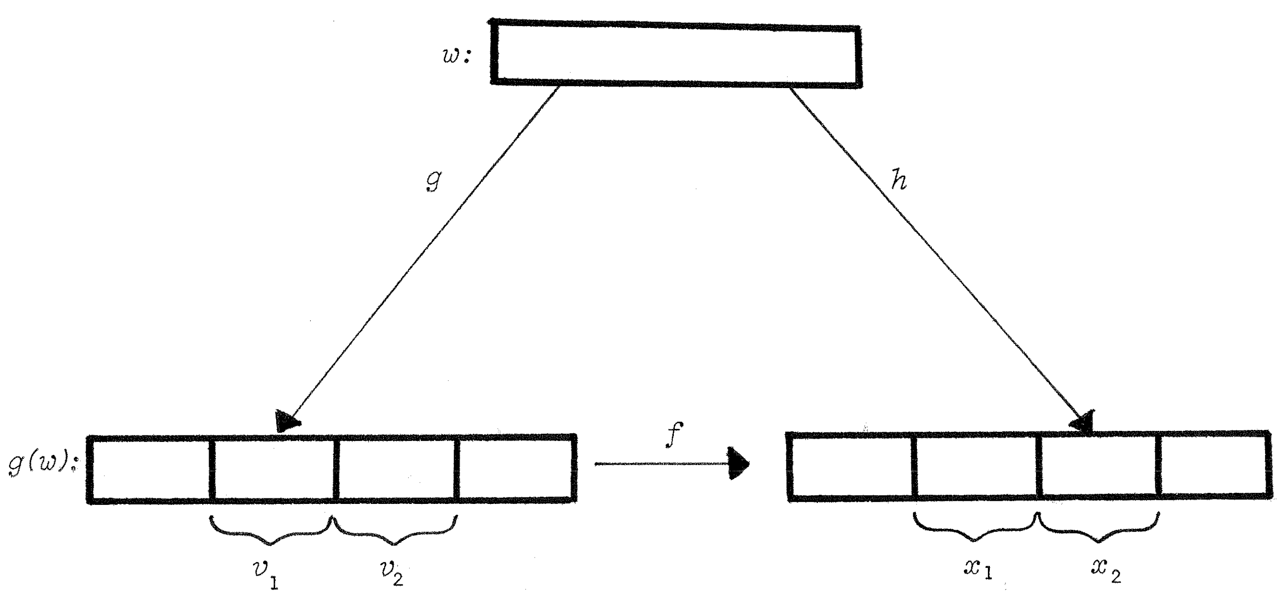
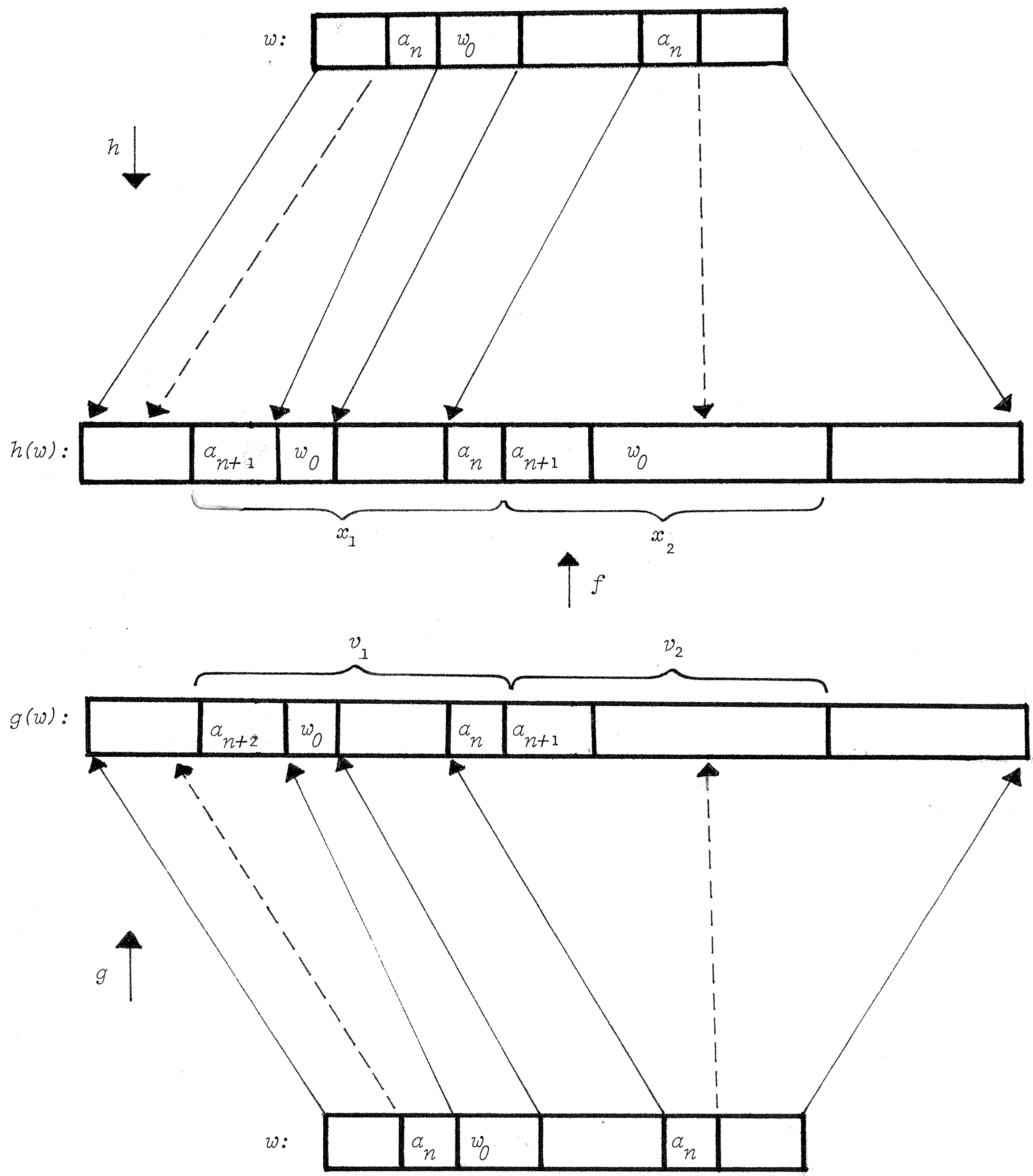


Figure 1



w:
a n
w 0
a n
h
h(w):
a n+1
w 0
a n
a n+1
w 0
x 1
x 2
f
v 1
v 2
g(w):
a n+2
w 0
a n
a n+1
g
w:
a n
w 0
a n

Figure 2

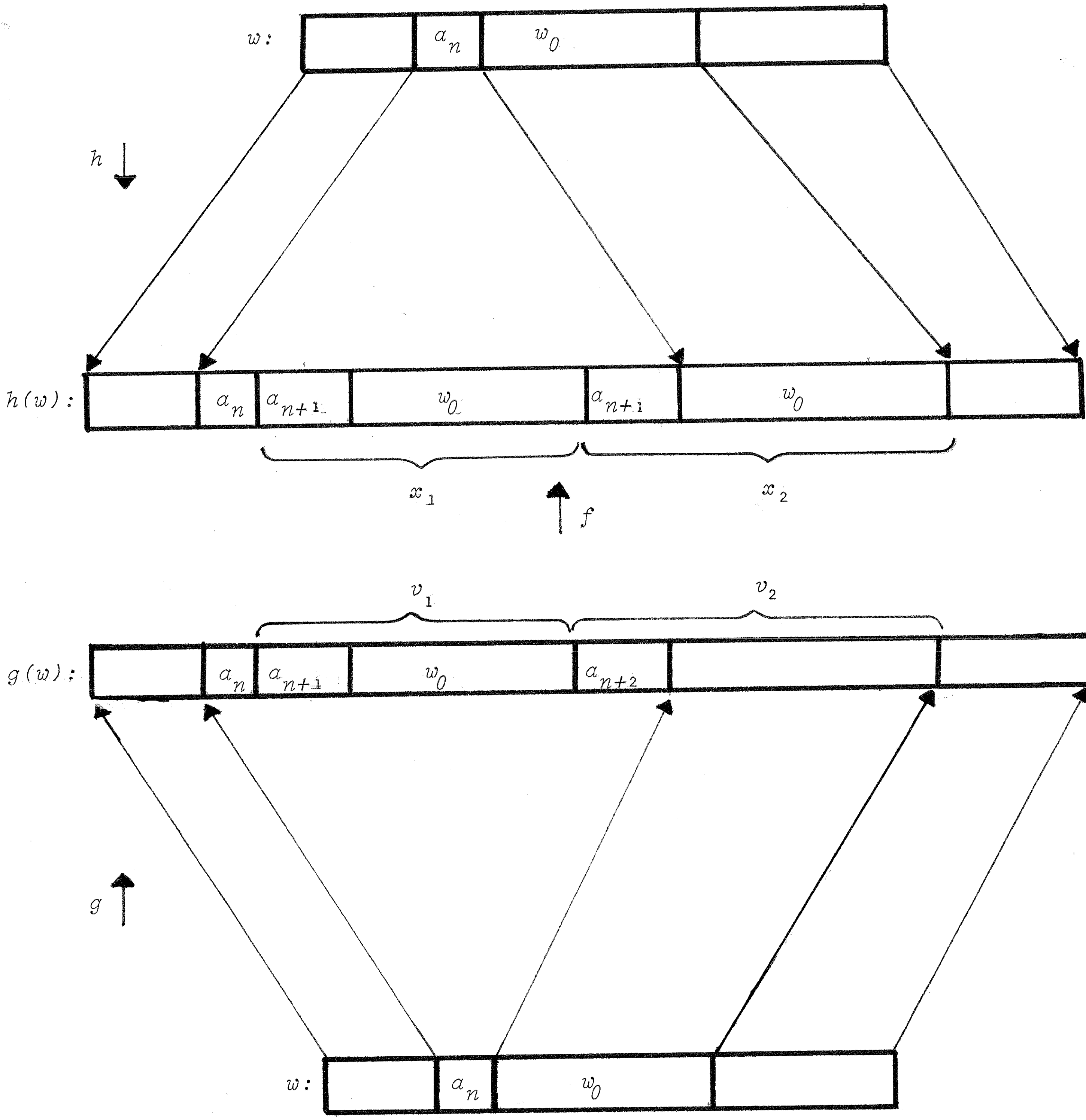


Figure 3