BASIC FORMULAS AND LANGUAGES
PART II. APPLICATIONS TO EOL SYSTEMS AND FORMS

by

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ABSTRACT

This paper (the second of two parts) settles the decidability status of several properties of derivations in EOL systems (forms). In particular we show that the so called "one-to-many simulation" among EOL forms is decidable, solving in this way an open problem from [6]. We use a more general mathematical framework, based on the theory of well-quasi-orders, developed in Part I of this paper ([2]).

INTRODUCTION

Analysis of derivations in various kinds of grammars constitutes a very important part of research in formal language theory (see, e.g., [3] and [8]). Such an analysis is very crucial within the theory of grammatical similarity (see, e.g., [6], [7] and [9]). In the first part of the paper we have developed a general mathematical framework, based on the theory of well-quasi-ordering, to deal with some (decision problems concerning) properties of derivations in EOL systems (forms). In this paper we apply results from [2] to settle the decidability status of several problems concerning the possibilities of simulation of one EOL form by another. In particular we show that the "one-to-many simulation" among EOL forms ([6]) is decidable. This result together with [1] demonstrates that both fundamental simulation lemmas from [6] ("one-to-many" and "many-to-one") are effective.

I. ONE-TO-MANY SIMULATION IN EOL FORMS

The general technique used in the proofs is the following. If we have to decide about a property P concerning two EOL systems $\overline{G}$ and G', i.e. decide whether or not $P(\overline{G},G')$, first a third EOL system is constructed which carries "enough information" about $\overline{G}$ and G'. Then basic languages (over $\theta \cup \Omega$) $\overline{K}_1$ and $\overline{K}_2$ are defined in a way that they "control the property P ". Then $K_1 = \overline{K}_1 \cap \theta^* \Omega^*$ and $K_2 = \overline{K}_2 \cap \theta^* \Omega^*$ yield (using finite substitutions φ and ψ originated from G) the sequences $\tau(K_1,K_2)$ and $\rho(K_1,K_2) = L_1,L_2,\ldots$ (see [2], Section III). Then it will be proved that $P(\overline{G},G')$ if and only if there exists a positive integer ℓ such that the axiom of G belongs to L_ℓ. Since the latter question is a decidable one (see [2], Theorem III.1), the effectiveness of all above constructions yields the decidability of the property P for any two EOL systems $\overline{G}$ and G'.

Before presenting our first decidability result concerning EOL systems we state the following definitions concerning derivations in an EOL system.

<u>Definition</u>. Let $G = (\Sigma,\varphi,\omega,\Delta)$ be an EOL system. Let $X,Y \subseteq \Sigma^*$ and let ℓ be a positive integer.

(1) Then we define the following sets.

$\mathcal{D}_G(X,Y,\ell) = \{D$: there exists a pair $(x,y) \in X \times Y$ such that D is a derivation of length ℓ in G starting from x and leading to $y\}$.

$\mathcal{D}_{ntG}(X,Y,\ell) = \{D \in \mathcal{D}_G(X,Y,\ell)$: each $x \in$ <u>itrace</u> D contains at least one nonterminal$\}$.

(2) If $m \geq 1$ and for $1 \leq i \leq m$, $D_i \in \mathcal{D}_G(x_i,\Sigma^*,\ell)$ where $x_i \in \Sigma^*$, then $\langle D_1D_2\ldots D_m\rangle$ denotes the derivation $D \in \mathcal{D}_G(x_1x_2\ldots x_m,\Sigma^*,\ell)$ the derivation tree (forest) of which results from the derivation trees (forests) representing $D_1,D_2,\ldots,D_m$ by putting them next to each other (in this order). □

The following definition formally describes the property we are dealing with.

Definition. Let $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ be two EOL systems such that $\overline{\Sigma} \subseteq \Sigma'$. $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$.

Then we define the property P_1 as follows.

$P_1(\overline{G},G')$

if and only if

there exists a positive integer ℓ such that for all $\alpha \underset{\overline{\varphi}}{\rightarrow} x$ there exists a $D \in \mathcal{D}_{nt\ G'}(\alpha,x,\ell)$. □

Then we have the following theorem.

Theorem I.1. For any two EOL systems $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ such that $\overline{\Sigma} \subseteq \Sigma'$, $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$ it is decidable whether or not $P_1(\overline{G},G')$.

Proof.

Let $\overline{G}$ and G' be as in the statement of the theorem. We will construct an EOL system $G = (\theta,\varphi,\omega_1,\Delta_1)$ such that $P_1(\overline{G},G')$ if and only if ω_1 satisfies a statement which will be proved to be decidable using our results concerning basic languages.

Construction of $G = (\theta,\varphi,\omega_1,\Delta_1)$.

Let $\overline{\varphi}$ be given by the following productions: $\pi_1,\pi_2,\ldots,\pi_s$, where $s \geq 1$. Then

$\theta = \{[\sigma,\pi_i,x] : \sigma \in \Sigma',\ 1 \leq i \leq s \text{ and } x \in \underline{\text{sub}}\ \overline{1}\ \overline{2} \ldots \overline{|\text{rhs}\ \pi_i|}\}$,

$\Delta_1 = \{[\sigma,\pi_i,x] \in \theta : \sigma \in \Delta'\}$, and

$\omega_1 = [\underline{\text{lhs}}\ \pi_1,\pi_1,\overline{1}\ \overline{2} \ldots \overline{|\underline{\text{rhs}}\ \pi_1|}][\underline{\text{lhs}}\ \pi_2,\pi_2,\overline{1}\ \overline{2} \ldots \overline{|\text{rhs}\ \pi_2|}]\ldots$
$[\underline{\text{lhs}}\ \pi_s,\pi_s,\overline{1}\ \overline{2} \ldots \overline{|\text{rhs}\ \pi_s|}]$.

φ is defined as follows.

(i) If $[\sigma,\pi_i,x] \in \theta$ and $\Lambda \in \varphi'(\sigma)$, then $\Lambda \in \varphi([\sigma,\pi_i,x])$.

(ii) If $[\sigma,\pi_i,x] \in \theta$ and $\alpha_1\alpha_2\ldots\alpha_k \in \varphi'(\sigma)$, $k \geq 1$ and for $1 \leq i \leq k$, $\alpha_i \in \Sigma'$, then $[\alpha_1,\pi_i,x_1][\alpha_2,\pi_i,x_2] \ldots [\alpha_k,\pi_i,x_k] \in \varphi([\sigma,\pi_i,x])$ whenever $x = x_1x_2\ldots x_k$.

(iii) $\varphi(\theta)$ contains no other elements. □

Hence intuitively speaking the EOL system G codes the following information. Let, e.g., $\pi_i : \alpha \to \alpha_1\alpha_2 \ldots \alpha_k$ be a production of $\overline{G}$, $k \geq 1$ and for $1 \leq i \leq k$, $\alpha_i \in \Sigma'$. Furthermore, let $[\sigma,\pi_i,\overline{j}_1\ldots\overline{j}_2] \in \theta$ where $1 \leq j_1 \leq j_2$. Then $[\sigma,\pi_i,\overline{j}_1..\overline{j}_2]$ indicates that we try to simulate in G' the production π_i of $\overline{G}$ and moreover we have derived the symbol $\sigma \in \Sigma'$ which "promises" to derive the subword $\alpha_{j_1} \ldots \alpha_{j_2}$ of <u>rhs</u> π_i. Hence inspecting ω_1 and φ, it must be rather clear that we try to simulate (in a parallel way) all productions of $\overline{G}$ using (coded versions of the) productions of G'.

<u>Construction of two basic languages K_1 and K_2.</u>

With each nonerasing production π of $\overline{G}$ we associate a basic formula. If π equals $\alpha \to \alpha_1\alpha_2 \ldots \alpha_{|\underline{rhs}\,\pi|}$ where for $1 \leq i \leq |\underline{rhs}\,\pi|$, $\alpha_i \in \overline{\Sigma}$ then $\Gamma_\pi(\xi) = \underline{me}([\alpha_1,\pi,\overline{1}],1,\xi) \wedge \underline{me}([\alpha_2,\pi,\overline{2}],1,\xi) \wedge \ldots \wedge \underline{me}([\alpha_{|\underline{rhs}\,\pi|},\pi,\overline{|\underline{rhs}\,\pi|}],1,\xi)$.
Let Π_Λ (Π respectively) denote the set of all (nonerasing) productions of $\overline{G}$ and $\theta' = \theta \setminus (\bigcup_{\pi\in\Pi} \underline{alph}\,\Gamma_\pi(\xi))$. Then define

$\Phi_1(\xi) = \bigwedge_{\pi\in\Pi} \Gamma_\pi(\xi) \wedge \bigwedge_{\sigma\in\theta'} \underline{ez}(\sigma,\xi)$, and

$\Phi_2(\xi) = \bigwedge_{\pi\in\Pi_\Lambda} \left(\bigvee_{(\sigma,x)\in X_\pi} \underline{me}([\sigma,\pi,x],1,\xi)\right) \wedge \bigwedge_{\sigma\in\theta} \underline{me}(\sigma,0,\xi)$, where for each $\pi \in \Pi_\Lambda$ $X_\pi = \{(\sigma,x) : \sigma \in \Sigma' \setminus \Delta',\ x \in \underline{sub}\ \overline{1}\ \overline{2} \ldots \overline{|\underline{rhs}\,\pi|}\}$. Finally let

$K_1 = L(\Phi_1(\xi))$ and $K_2 = L(\Phi_2(\xi))$. □

Hence intuitively speaking a word $w \in \theta^*$ belongs to K_1 if and only if it contains all letters of

$\{[\sigma,\pi,\overline{i}] : \pi \in \Pi,\ 1 \leq i \leq |\underline{rhs}\,\pi|$ and σ is the i'th letter of <u>rhs</u> $\pi\}$

and no other letters. A word $w \in \theta^*$ belongs to K_2 if and only if it contains for each $\pi \in \Pi_\Lambda$ at least one letter $[\sigma,\pi,x] \in \theta$ with σ a nonterminal letter of G'.

<u>Construction of two sequences of languages.</u>

We now apply the basic construction of Part I of our paper (see [2], Section III) with φ,θ,K_1,K_2 as above and $\Omega = \{\Lambda\}$ to get two sequences of languages $\tau(K_1,K_2)$ and $\rho(K_1,K_2) = L_1,L_2,\ldots$. □

Now we claim the following.

Claim I.1.

$P_1(\overline{G},G')$

if and only if

there exists a positive integer ℓ such that $\omega_1 \in L_\ell$.

Proof of Claim I.1.

The only if-part is trivial.

To prove the if-part, assume $\omega_1 \in L_\ell$ for some positive integer ℓ. Then Lemma III.2 from [2] implies that $\omega_1 \in \theta^*$ and there exists a derivation $D : \omega_1 \underset{\varphi}{\Rightarrow} u_1 \underset{\varphi}{\Rightarrow} u_2 \underset{\varphi}{\Rightarrow} \dots \underset{\varphi}{\Rightarrow} u_\ell$ such that $u_\ell \in K_1$ and $u_i \in K_2$ for $1 \leq i < \ell$. Inspecting the form of $\Phi_2(\xi)$ and φ, it suffices to prove that $u_\ell = y$ with $y = y_1 y_2 \dots y_s$ where for $1 \leq i \leq s$,

$y_i = \Lambda$ if $\underline{\text{rhs}}\ \pi_i = \Lambda$, and

$y_i = [\alpha_{i,1},\pi_i,\overline{1}][\alpha_{i,2},\pi_i,\overline{2}] \dots [\alpha_{i,|\underline{\text{rhs}}\pi_i|},\pi_i,\overline{|\underline{\text{rhs}}\ \pi_i|}]$ if

$\underline{\text{rhs}}\ \pi_i = \alpha_{i,1}\ \alpha_{i,2} \dots \alpha_{i,|\underline{\text{rhs}}\ \pi_i|} \neq \Lambda$ and for $1 \leq j \leq |\text{rhs}\ \pi_i|, \alpha_{i,j} \in \overline{\Sigma}$.

The fact that $u_\ell = y$ immediately follows from the following three observations:

(i) the definition of $\Phi_1(\xi)$ guarantees $\underline{\text{alph}}\ u_\ell = \underline{\text{alph}}\ y$,

(ii) the definition of φ guarantees every letter occurs only once in u_ℓ,

(iii) the definitions of φ and ω_1 guarantee all letters occur in u_ℓ precisely in the same order as they occur in y.

Hence Claim I.1. holds. □

Since G and the basic languages K_1, K_2 can be effectively constructed, the above claim together with Theorem III.1 from [2] yield the theorem. □

We will consider now the so called "one-to-many simulation" among EOL systems (forms). The decidability status of the problem is the basic open problem concerning the simulation of one EOL form by another (see, e.g., [1] and [6]). We demonstrate that the problem is decidable. The solution is based on a construction analogous to the one of Theorem I.1.

Definition. ("One-to-many simulation", see [6])

Let $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ be two reduced EOL systems such that $\overline{\Sigma} \subseteq \Sigma'$, $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$.

Then we define the property P_2 as follows.

$P_2(\overline{G},G')$

if and only if

there exists a positive integer ℓ such that for all $\alpha \underset{\varphi}{\rightarrow} x$ there exists a $D \in \mathcal{D}_{G'}(\alpha_1,x,\ell)$ such that if $\omega' \underset{G'}{\overset{*}{\Rightarrow}} y\alpha z$ then for all $(D_1,D_2) \in \mathcal{D}_{G'}(y,\Sigma'^*,\ell) \times \mathcal{D}_{G'}(z,\Sigma'^*,\ell)$, $<D_1DD_2> \in \mathcal{D}_{ntG'}(y\alpha z,\Sigma'^*,\ell)$. □

Theorem I.2. For any two reduced EOL systems $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ such that $\overline{\Sigma} \subseteq \Sigma'$, $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$ it is decidable whether of not $P_2(\overline{G},G')$.

Proof.

We proceed as in the proof of Theorem I.1 and we will use the notations and definitions stated there.

Construction of $H = (\theta \cup \Omega, \varphi \cup \psi, \omega_1\omega_2, \Delta_1 \cup \Delta_2)$.

Let $\theta,\varphi,\omega_1,\Delta_1,\Pi$ and Π_Λ be as in the proof of Theorem I.1. Let

$\Omega = \{[\sigma,\pi,Z] : \sigma \in \Sigma',\ \pi \in \Pi_\Lambda,\ Z \in \underline{sur}_{G'}(\underline{lhs}\ \pi)\}$,

$\Delta_2 = \{ [\sigma,\pi,Z] \in \Omega : \sigma \in \Delta'\}$, and

$\omega_2 = \omega_{\pi_1}\omega_{\pi_2}\ldots\omega_{\pi_s}$ where for $1 \leq i \leq s$,

(i) if $\emptyset \in \underline{sur}_{G'}(\underline{lhs}\ \pi_i)$ then $\omega_{\pi_i} = \Lambda$,

(ii) if $\emptyset \notin \underline{sur}_{G'}(\underline{lhs}\ \pi_i) = \{Z_1,Z_2,\ldots,Z_t\}$, $t \geq 1$ with for $1 \leq j \leq t$, $Z_j = \{\sigma_{j,1},\sigma_{j,2},\ldots,\sigma_{j,\ell_j}\}$, $\ell_j \geq 1$, then

$\omega_{\pi_i} = [\sigma_{1,1},\pi_i,Z_1][\sigma_{1,2},\pi_i,Z_1]\ldots[\sigma_{1,\ell_1},\pi_i,Z_1][\sigma_{2,1},\pi_i,Z_2][\sigma_{2,2},\pi_i,Z_2]\ldots$

$[\sigma_{2,\ell_2},\pi_i,Z_2]\ \ .\ \ .\ \ .\ \ [\sigma_{t,1},\pi_i,Z_t][\sigma_{t,2},\pi_i,Z_t]\ldots[\sigma_{t,\ell_t},\pi_i,Z_t]$.

The finite subsitution ψ is defined as follows.

(i) If $[\sigma,\pi,Z] \in \Omega$ and $\Lambda \in \varphi'(\sigma)$, then $\Lambda \in \psi([\sigma,\pi,Z])$.

(ii) If $[\sigma,\pi,Z] \in \Omega$ and $\alpha_1\alpha_2\ldots\alpha_k \in \varphi'(\sigma)$, $k \geq 1$ and for $1 \leq i \leq k$, $\alpha_i \in \Sigma'$, then $[\alpha_1,\pi,Z][\alpha_2,\pi,Z]\ldots[\alpha_k,\pi,Z] \in \psi([\sigma,\pi,Z])$.

(iii) $\psi(\Omega)$ contains no other elements. □

Intuitively speaking for each production π, ω_π codes the necessary information concerning the possible surroundings of $\underline{\text{lhs}}\ \pi$; ψ simulates the effect of applying production of φ' to the first argument of an element of Ω, but "preserving" the "surrounding information".

Construction of the languages $\tilde{K}_1$ and $\tilde{K}_2$.

Let $\Phi_1(\xi)$ and $\Phi_2(\xi)$ be as in the proof of Theorem I.1. Let

$$\tilde{\Phi}_1(\xi) = \Phi_1(\xi) \wedge \bigwedge_{\sigma\in\Omega} \underline{\text{me}}(\sigma,0,\xi), \text{ and}$$

$$\tilde{\Phi}_2(\xi) = \Big(\bigwedge_{(\pi,Z)\in X_1} \Big(\bigvee_{(\sigma,x)\in X_\pi} \underline{\text{me}}([\sigma,\pi,x],1,\xi) \vee \bigvee_{\sigma\in\Sigma'\setminus\Delta'} \underline{\text{me}}([\sigma,\pi,Z],1,\xi)\Big)\Big)$$

$$\wedge \Big(\bigwedge_{\pi\in X_2} \Big(\bigvee_{(\sigma,x)\in X_\pi} \underline{\text{me}}([\sigma,\pi,x],1,\xi)\Big)\Big)$$

$$\wedge \Big(\bigwedge_{\sigma\in\Theta\cup\Omega} \underline{\text{me}}(\sigma,0,\xi)\Big), \text{ where}$$

$X_1 = \{(\pi,z) : \pi \in \Pi_\Lambda,\ \emptyset \notin \underline{\text{sur}}_{G'}(\underline{\text{lhs}}\ \pi) \text{ and } Z \in \underline{\text{sur}}_{G'}(\underline{\text{lhs}}\ \pi)\}$,

$X_2 = \{\pi \in \Pi_\Lambda : \emptyset \in \underline{\text{sur}}_{G'}(\underline{\text{lhs}}\ \pi)$, and for each $\pi \in \Pi_\Lambda$

$X_\pi = \{(\sigma,x) : \sigma \in \Sigma' \setminus \Delta' \text{ and } x \in \underline{\text{sub}}\ \bar{1}\ \bar{2} \ldots \overline{|\text{rhs}\ \pi|}\}$.

Finally $\bar{K}_1 = L(\tilde{\Phi}_1(\xi))$, $\bar{K}_2 = L(\tilde{\Phi}_2(\xi))$, $\tilde{K}_1 = \Theta^*\Omega^* \cap \bar{K}_1$ and $\tilde{K}_2 = \Theta^*\Omega^* \cap \bar{K}_2$.

Observe that $\bar{K}_1$ and $\bar{K}_2$ are Ω-positive languages. □

Hence intuitively speaking a word $w \in \Theta^*\Omega^*$ belongs to $\tilde{K}_1$ if and only if its Θ-part belongs to K_1 (see the proof of Theorem I.1). A word $w \in \Theta^*\Omega^*$ belongs to K_2 if and only if the following conditions hold.

(i) For every $\pi \in \Pi_\Lambda$ such that $\emptyset \in \underline{\text{sur}}_{G'}(\underline{\text{lhs}}\ \pi)$ (thus $\omega' \overset{*}{\underset{G'}{\Rightarrow}} \underline{\text{lhs}}\ \pi$), w contains at least one nonterminal letter $[\sigma,\pi,x] \in \Theta$.

(ii) For every $\pi \in \Pi_\Delta$ such that $\emptyset \notin \underline{\text{sur}}_{G'}(\underline{\text{lhs}}\ \pi)$ and every possible $Z \in \underline{\text{sur}}_{G'}(\underline{\text{lhs}}\ \pi)$ (thus $\omega' \overset{*}{\underset{G'}{\Rightarrow}} u(\underline{\text{lhs}}\ \pi)v$ with $(\underline{\text{alph}}\ u(\underline{\text{lhs}}\ \pi)v) \setminus \{\underline{\text{lhs}}\ \pi\} = Z$), w contains at least one nonterminal letter $[\sigma,\pi,x] \in \theta$ or at least one nonterminal letter $[\sigma,\pi,Z] \in \Omega$.

Construction of two sequences of languages.

We now apply the basic construction of Part I of our paper (see [2], Section III) with $\varphi,\psi,\theta,\Omega,\tilde{K}_1$ and $\tilde{K}_2$ as above to get two sequences of languages $\rho(\tilde{K}_1,\tilde{K}_2)$ and $\tau(\tilde{K}_1,\tilde{K}_2) = L_1,L_2 \ldots,$□

Now we claim the following.

Claim I.2.

$P_2(\overline{G},G')$

if and only if

there exists a positive integer ℓ such that $\omega_1\omega_2 \in L_\ell$.

Proof of Claim I.2.

The only if-part is trivial.

To prove the if-part assume $\omega_1\omega_2 \in L_\ell$ for some positive integer ℓ. Then Lemma III.2 from [2] implies the existence of a derivation

$D : \omega_1 \underset{\varphi}{\Rightarrow} u_1 \underset{\varphi}{\Rightarrow} u_2 \underset{\varphi}{\Rightarrow} \ldots \underset{\varphi}{\Rightarrow} u_\ell$ such that for all derivations

$D' : \omega_2 \underset{\psi}{\Rightarrow} v_1 \underset{\psi}{\Rightarrow} v_2 \underset{\psi}{\Rightarrow} \ldots \underset{\psi}{\Rightarrow} v_\ell$, $u_\ell v_\ell \in \tilde{K}_1$ and $u_i v_i \in \tilde{K}_2$ for $1 \leq i < \ell$.

Again as in Theorem I.1 we can prove $u_\ell = y$ (y is defined as in the proof of Claim I.1). Then inspecting $\tilde{\Phi}_2(\xi)$, φ and ψ, the claim immediately follows. □

The effectiveness of all above constructions then yields the theorem. □

We end the section by considering a property closely related to P_2 which is defined as follows.

Definition. Let $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ be two reduced EOL systems such that $\overline{\Sigma} \subseteq \Sigma'$, $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$. Then

$P_2'(\overline{G},G')$

if and only if

there exists a positive integer ℓ such that for all $\alpha \underset{\varphi}{\to} x$ there exists a $D \in \mathcal{D}_{G'}(\alpha,x,\ell)$ such that if $\overline{\omega} \overset{*}{\underset{\overline{G}}{\Rightarrow}} y\alpha z$ then for all

$(D_1,D_2) \in \mathcal{D}_{G'}(y,\Sigma'^*,\ell) \times \mathcal{D}_{G'}(z,\Sigma'^*,\ell)$, $\langle D_1DD_2\rangle \in \mathcal{D}_{ntG'}(y\alpha z,\Sigma'^*,\ell)$. □

Observe that clearly $P_2(\overline{G},G')$ implies $P_2'(\overline{G},G')$, but in general the converse does not hold. We immediately get the following result.

Corollary I.1. For any two reduced EOL systems $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ such that $\overline{\Sigma} \subseteq \Sigma$, $\overline{\Delta} \subseteq \Delta$ and $\overline{\omega} = \omega'$ it is decidable whether or not $P_2'(\overline{G},G')$.

Proof.

Replace $\text{sur}_{G'}(\text{lhs } \pi)$ by $\text{sur}_{\overline{G}}(\text{lhs } \pi)$ in the proof of Theorem I.2. □

II. CONTROLLED DERIVATIONS

Before discussing next "concrete" decidability questions concerning EOL systems, we present some general results concerning derivations in an EOL system "controlled by a sequence of basic languages" which are also interesting on its own.

First we need the following definition which makes the notion "controlled by a sequence of basic languages" precise.

Definition.

(i) Let $G = (\Sigma,\varphi,\omega,\Delta)$ be an EOL system and

$\tau_0 : K_1, K_1', K_2, K_2', \ldots, K_s, K_s'$, $s \geq 1$ a finite sequence of basic languages over Σ.

Then

$$D : \omega \underset{G}{\Rightarrow} u_{1,1} \underset{G}{\Rightarrow} u_{1,2} \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} u_{1,i_1} \underset{G}{\Rightarrow} u_{2,1} \underset{G}{\Rightarrow} u_{2,2} \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} u_{2,i_2} \underset{G}{\Rightarrow} \cdot\ \cdot\ \cdot \underset{G}{\Rightarrow}$$
$$u_{s,1} \underset{G}{\Rightarrow} u_{s,2} \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} u_{s,i_s}$$

, $i_j \geq 1$ for $1 \leq j \leq s$ is called a $(K_1, K_1', K_2, K_2', \ldots, K_s, K_s')$-derivation in G or a τ_0-derivation in G if

for $1 \leq j \leq s$, $u_{j,i_j} \in K_j'$, and

for $1 \leq j \leq s$, $1 \leq p < i_j$, $u_{j,p} \in K_j$.

(ii) Let $m \geq 1$, $s \geq 1$ and for $1 \leq j \leq m$ let $G^{(j)} = (\Sigma^{(j)},\varphi^{(j)},\omega^{(j)},\Delta^{(j)})$ be EOL systems such that their alphabets are mutually disjoint. For $1 \leq j \leq m$, let $\tau_j : K_{j,1}, K_{j,1}', K_{j,2}, K_{j,2}', \ldots, K_{j,s}, K_{j,s}'$ be a sequence of basic languages over $\Sigma^{(j)}$. Further, for $1 \leq j \leq m$ let

$$D_j : \omega^{(j)} \underset{G^{(j)}}{\Rightarrow} u_{j,1,1} \underset{G^{(j)}}{\Rightarrow} u_{j,1,2} \underset{G^{(j)}}{\Rightarrow} \cdots \underset{G^{(j)}}{\Rightarrow} u_{j,1,i_{j,1}} \underset{G^{(j)}}{\Rightarrow} u_{j,2,1} \underset{G^{(j)}}{\Rightarrow}$$
$$u_{j,2,2} \underset{G^{(j)}}{\Rightarrow} \cdots \underset{G^{(j)}}{\Rightarrow} u_{j,2,i_{j,2}} \underset{G^{(j)}}{\Rightarrow} \cdot\ \cdot\ \cdot \underset{G^{(j)}}{\Rightarrow} u_{j,s,1} \underset{G^{(j)}}{\Rightarrow} u_{j,s,2} \underset{G^{(j)}}{\Rightarrow} \cdots$$
$$\underset{G^{(j)}}{\Rightarrow} u_{j,s,i_{j,s}}$$

be a τ_j-derivation such that

for $1 \leq p \leq s$, $u_{j,p,i_{j,p}} \in K_{j,p}'$ and

for $1 \leq p \leq s$, $1 \leq q < i_{j,p}$, $u_{j,p,q} \in K_{j,p}$.

Then $(D_1,D_2,\ldots,D_m)$ is called $(\tau_1,\tau_2,\ldots,\tau_m)$-controlled if for $1 \le j \le m$, $1 \le p \le s$, $i_{j,p} = i_{1,p}$. □

Then we have the following two results.

Lemma II.1. For any EOL system $G = (\Sigma,\varphi,\omega,\Delta)$ and a sequence τ_0 of basic languages over $\Sigma,\tau_0 : K_1,K_1',K_2,K_2',\ldots,K_s,K_s'$, $s \ge 1$, it is decidable whether or not there exists a τ_0-derivation in G.

Proof.

Let G and τ_0 be as in the statement of the lemma.

Consider the following sequence of basic languages (which can be effectively computed, see Corollary III.1 from [2]).

$M_s = g_\Sigma(\varphi,K_s,K_s')$, and for $1 \le i < s$,

$M_{s-i} = g_\Sigma(\varphi,K_{s-i},K_{s-i}' \cap M_{s-i+1})$.

Then obviously there exists a τ_0-derivation in G if and only if $\omega \in M_1$.

Since clearly it is decidable whether or not $\omega \in M_1$, the lemma holds. □

Lemma I.2. Let $m \ge 1$, $s \ge 1$ and for $1 \le j \le m$ let $G^{(j)} = (\Sigma^{(j)},\varphi^{(j)},\omega^{(j)},\Delta^{(j)})$ be EOL systems such that their alphabets are mutually disjoint. For $1 \le j \le m$ let

$\tau_j : K_{j,1},K_{j,1}',K_{j,2},K_{j,2}',\ldots,K_{j,s},K_{j,s}'$ be a sequence of basic languages over $\Sigma^{(j)}$.

Then it is decidable whether or not there exist D_j, τ_j-derivations in $G^{(j)}$, $1 \le j \le m$, such that $(D_1,D_2,\ldots,D_m)$ is $(\tau_1,\tau_2,\ldots,\tau_m)$-controlled.

Proof.

Using the above notations define the EOL system $H = (\Sigma,\varphi,\omega,\Delta)$ where

$$\Sigma = \bigcup_{j=1}^{m} \Sigma^{(j)}, \quad \Delta = \bigcup_{j=1}^{m} \Delta^{(j)}, \quad \varphi = \bigcup_{j=1}^{m} \varphi^{(j)}, \quad \omega = \omega^{(1)}\omega^{(2)} \ldots \omega^{(m)}.$$

Assume that for $1 \le j \le m$, $1 \le p \le s$, $K_{j,p} = L(\Phi_p^{(j)}(\xi))$ and $K_{j,p}' = L(\Psi_p^{(j)}(\xi))$ where $\Phi_p^{(j)}(\xi)$ and $\Psi_p^{(j)}(\xi)$ are basic formulas. Then for $1 \le p \le s$ let

$K_p = L(\bigwedge_{j=1}^{m} \Phi_p^{(j)}(\xi))$ and $K_p' = L(\bigwedge_{j=1}^{m} \Psi_p^{(j)}(\xi))$. Then define the following sequence of basic languages (which can be effectively computed, see Corollary III.1 from [2]).

$M_s = g_\Sigma(\varphi, K_s, K_s')$, and for $1 \le p < s$,

$M_{s-p} = g_\Sigma(\varphi, K_{s-p}, K'_{s-p} \cap M_{s-p+1})$.

Obviously there exist D_j, τ_j-derivations in $G^{(j)}$, $1 \le j \le m$, such that $(D_1, D_2, \ldots, D_m)$ is $(\tau_1, \tau_2, \ldots, \tau_m)$-controlled if and only if $\omega \in M_1$. Since clearly it is decidable whether or not $\omega \in M_1$, the lemma holds. □

We will prove now two decidability results concerning derivations in EOL systems which make use of the two previous lemmas. These properties are formally defined now.

Definition.

(i) Let $G = (\Sigma, \varphi, \omega, \Delta)$ be an EOL system, $x \in \Sigma^+$, $y \in \Sigma^*$ and ℓ a positive integer.

Then we define the property P_3 as follows.

$P_3(G, x, y, \ell)$

if and only if

$\omega \overset{*}{\underset{G}{\Rightarrow}} uxv$ for some $u, v \in \Sigma^*$ and there exist $D \in \mathcal{D}_G(x, y, \ell)$, $D_1 \in \mathcal{D}_G(u, \Sigma^*, \ell)$, $D_2 \in \mathcal{D}_G(v, \Sigma^*, \ell)$ such that $\langle D_1 D D_2 \rangle \in \mathcal{D}_{ntG}(uxv, \Sigma^*, \ell)$.

(ii) Let $\overline{G} = (\overline{\Sigma}, \overline{\varphi}, \overline{\omega}, \overline{\Delta})$ and $G' = (\Sigma', \varphi', \omega', \Delta')$ be two EOL systems such that $\overline{\Sigma} \subseteq \Sigma'$, $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$.

Then we define the property P_4 as follows.

$P_4(\overline{G}, G')$

if and only if

there exist a positive integer ℓ such that for all $\alpha \underset{\overline{\varphi}}{\rightarrow} x$, $P_3(G', \alpha, x, \ell)$. □

Then we have the following results.

Theorem II.1. For any EOL system $\bar{G} = (\bar{\Sigma},\bar{\varphi},\bar{\omega},\bar{\Delta})$, $x \in \bar{\Sigma}^{+}$, $y \in \bar{\Sigma}^{*}$, it is decidable whether or not there exists a positive integer ℓ such that $P_3(\bar{G},x,y,\ell)$.

Proof.

Let $x = \alpha_1\alpha_2 \dots \alpha_n$, $y = \beta_1\beta_2 \dots \beta_m$, $n > 0$, $m \geq 0$ where for $1 \leq i \leq n$, $\alpha_i \in \bar{\Sigma}$ and for $1 \leq i \leq m$, $\beta_i \in \bar{\Sigma}$.

First construct the EOL system $G = (\Sigma,\varphi,S,\Delta)$ where S is a new symbol, $\Sigma = \bar{\Sigma} \cup \{S\}$, $\Delta = \bar{\Delta}$, $\varphi(\alpha) = \bar{\varphi}(\alpha)$ if $\alpha \in \bar{\Sigma}$ and $\varphi(S) = \{\bar{\omega}\}$.

Construction of the EOL system $G' = (\Sigma',\varphi', S',\Delta')$.

Let $\Sigma' = \Sigma \cup \tilde{\Sigma}$, $\tilde{\Sigma} \cap \Sigma = \emptyset$,

$\tilde{\Sigma} = \{[\sigma,i,w,w'] : \sigma \in \Sigma,\ i \in \{1,2\},\ w \in \underline{\text{sub}}\ \bar{1}\,\bar{2}\dots\bar{n} \text{ and } w' \in \underline{\text{sub}}\ \bar{1}\,\bar{2}\dots\bar{m}\}$,

$\Delta' = \Delta \cup \{[\sigma,i,w,w'] \in \tilde{\Sigma} : \sigma \in \Delta\}$, and $S' = [S,1,x,y]$.

φ' is defined as follows.

(i) If $\sigma \in \Sigma$, then $\varphi'(\sigma) = \varphi(\sigma)$.

(ii) If $[\sigma,i,w,w'] \in \tilde{\Sigma}$ and $\Lambda \in \varphi(\sigma)$ then also $\Lambda \in \varphi'([\sigma,i,w,w'])$.

(iii) If $[\sigma,i,w,w'] \in \tilde{\Sigma}$ and $\gamma_1\gamma_2 \dots \gamma_k \in \varphi(\sigma)$, $k \geq 1$ and $\gamma_j \in \Sigma$ for $1 \leq j \leq k$, we have to consider several cases.

(iii.1) $i = 1$.

(iii.1.1) If $w = \bar{1}\,\bar{2}\dots\bar{n}$ then

$\gamma_1\gamma_2\dots\gamma_{q-1}[\gamma_q,i',w_q,w'_q][\gamma_{q+1},i',w_{q+1},w'_{q+1}]\dots[\gamma_r,i',w_r,w'_r]\gamma_{r+1}\dots\gamma_k \in \varphi'([\sigma,i,w,w'])$, where $i' \in \{1,2\}$, $1 \leq q \leq r \leq k$, $w = w_q w_{q+1}\dots w_r$, $w' = w'_q w'_{q+1}\dots w'_r$ and, for $q \leq j \leq r$, $w_j \in \{\bar{1},\bar{2},\dots,\bar{n}\}^*$ and $w'_j \in \{\bar{1},\bar{2},\dots,\bar{m}\}^*$.

(iii.1.2) If $w = \bar{1}\,\bar{2}\dots\bar{s}$ for some $1 \leq s < n$ then do as in (iii.1.1) except that now $1 \leq q \leq r = k$.

(iii.1.3) If $w = \bar{s}\,\overline{(s+1)}\dots\bar{n}$ with $1 < s \leq n$ then do as in (iii.1.1) except that now $1 = q \leq r \leq k$.

(iii.1.4) If $w = \bar{s}\,\overline{(s+1)}\dots\bar{t}$ with $1 < s \leq t < n$ or if $w = \Lambda$ then do as in (iii.1.1) except that now $1 = q \leq r = k$.

(iii.2) $i = 2$.

Do as in (iii.1.1) except that now always $i' = 2$ and $1 = q \leq r = k$. □

The idea behind the construction of G' is to simulate derivations of G and in this simulation the symbols of $\tilde{\Sigma}$ have the following meaning. Let, e.g., $1 \leq s \leq t \leq n$, $1 \leq s' \leq t' \leq m$ and $[\sigma,1,\overline{s}\ldots\overline{t},\overline{s'}\ldots\overline{t'}] \in \tilde{\Sigma}$, then this symbol codes the following information: we try to simulate $S \underset{G}{\Rightarrow} \overline{\omega} \overset{*}{\underset{G}{\Rightarrow}} uxv$ and we have derived the symbol σ which "promises" to derive the subword $\alpha_s\ldots\alpha_t$ of x and later on the subword $\beta_{s'}\ldots\beta_{t'}$ of y. If $[\sigma,2,w,\overline{s'}\ldots\overline{t'}] \in \tilde{\Sigma}$ (s' and t' as above), then this symbol codes the following information: we try to simulate $x \overset{+}{\underset{G}{\Rightarrow}} y$ and we have derived the symbol σ which "promises" to derive the subword $\beta_{s'}\ldots\beta_{t'}$ of y.

Construction of a sequence of basic languages.

We will construct now a sequence of basic languages to "control" the derivations in G'.

Let $\Sigma_1 = \{[\alpha_i,2,\overline{i},w'] \in \tilde{\Sigma} : i \in \{1,2,\ldots,n\}\}$,

$\Sigma_2 = \{[\sigma,i,w,w'] \in \tilde{\Sigma} : i = 2\}$, and

$\Sigma_3 = \{[\sigma,2,\overline{i},\overline{j}] \in \tilde{\Sigma} : i \in \{1,2,\ldots,n\},\ j \in \{1,2,\ldots,m\}\}$ if $m > 0$.

Then

$$\Phi_1(\xi) = \Big(\bigwedge_{\sigma\in\Sigma_2} \underline{ez}(\sigma,\xi)\Big) \wedge \Big(\bigwedge_{\sigma\in\Sigma_1} \underline{me}(\sigma,0,\xi)\Big),$$

$$\Psi_1(\xi) = \bigwedge_{i=1}^{n} \Big(\bigvee_{w\in\underline{sub}\,\overline{1}\,\overline{2}\ldots\overline{m}} \underline{me}([\alpha_i,2,\overline{i},w],1,\xi)\Big)$$

$$\wedge \bigwedge_{\sigma\in\tilde{\Sigma}\setminus\Sigma_1} \underline{ez}(\sigma,\xi)$$

$$\wedge \bigwedge_{\sigma\in\Sigma'} \underline{me}(\sigma,0,\xi),$$

$$\Phi_2(\xi) = \Big(\bigvee_{\sigma\in\Sigma'\setminus\Delta'} \underline{me}(\sigma,1,\xi)\Big) \wedge \Big(\bigwedge_{\sigma\in\Sigma} \underline{me}(\sigma,0,\xi)\Big), \text{ and}$$

$$\Psi_2(\xi) = \bigwedge_{j=1}^{m} \Big(\bigvee_{i=1}^{n} \underline{me}([\beta_j,2,\overline{i},\overline{j}],1,\xi)\Big)$$

$$\wedge \bigwedge_{\sigma\in\tilde{\Sigma}\setminus\Sigma_3} \underline{ez}(\sigma,\xi)$$

$$\wedge \bigwedge_{\sigma\in\Sigma'} \underline{me}(\sigma,0,\xi) \qquad \text{if } m > 0\text{; otherwise}$$

$$\Psi_2(\xi) = \left(\bigwedge_{\sigma\in\tilde{\Sigma}} \underline{ez}(\sigma,\xi)\right) \wedge \left(\bigwedge_{\sigma\in\Sigma'} \underline{me}(\sigma,0,\xi)\right).$$

Finally for $i = 1,2$ let $K_i = L(\Phi_i(\xi))$ and $K_i' = L(\Phi_i(\xi))$. □

Clearly if there exists a (K_1,K_1',K_2,K_2') - derivation in G', one can easily verify that $P_3(\overline{G},x,y,\ell)$ for some positive integer ℓ. Conversely, if $P_3(\overline{G},x,y,\ell)$ for some positive integer ℓ, then a (K_1,K_1',K_2,K_2') - derivation in G' can be constructed. Hence from Lemma II.2 the theorem follows. □

Theorem II.2. For any two EOL systems $\overline{G} = (\overline{\Sigma},\overline{\varphi},\overline{\omega},\overline{\Delta})$ and $G' = (\Sigma',\varphi',\omega',\Delta')$ such that $\overline{\Sigma} \subseteq \Sigma'$, $\overline{\Delta} \subseteq \Delta'$ and $\overline{\omega} = \omega'$, it is decidable whether or not $P_4(\overline{G},G')$.

Proof.

Let G and G' be as in the statement of the theorem and let S be a new symbol. Then define $\overline{G}_1 = (\overline{\Sigma}_1,\overline{\varphi}_1,S,\overline{\Delta})$ and $G_1' = (\Sigma_1',\varphi_1',S,\Delta')$ where $\overline{\Sigma}_1 = \overline{\Sigma} \cup \{S\}$, $\Sigma_1' = \Sigma' \cup \{S\}$. For each $\alpha \in \overline{\Sigma}$, $\overline{\varphi}_1(\alpha) = \overline{\varphi}(\alpha)$ and for each $\alpha \in \Sigma'$, $\varphi_1'(\alpha) = \varphi'(\alpha)$. Furthermore $\overline{\varphi}_1(S) = \varphi_1'(S) = \{S,\overline{\omega}\}$. Then a combination of the arguments of the proofs of Theorem II.1 and Lemma II.2 yields the theorem. □

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