BASIC FORMULAS AND LANGUAGES
PART I. THE THEORY

by

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CU-CS-191-80

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ABSTRACT

A general mathematical framework to deal with (the decidability status of) properties of derivations in EOL systems (forms) is developed. It is based on the theory of well-quasi-orders. This paper (the first of two parts) deals with the mathematical theory of the proposed approach.

INTRODUCTION

Analysis of derivations in various kinds of grammars constitutes a very important research area within formal language theory (see, e.g., [2] and [7]). This analysis becomes very cricial in the theory of grammatical similarity (see, e.g., [5], [6] and [8]). In particular various decision problems concerning comparability of various language families rely heavily on decision problems concerning the underlying (master) grammars, which, in the framework of L forms, is very well illustrated in [5]. The effectiveness status of two very basic simulation lemmas is left open there. This illustrates very well the general situation: we simply do not have yet general tools to deal with (decision problems concerning) the structure of derivations in EOL systems. There is certainly the need to develop mathematical tools to deal with this problem area.

In this paper we develop a framework to deal with the decision status of some properties of derivations in EOL systems. The paper is divided in two parts. The first part develops the mathematical theory of our approach which is based essentially on the theory of well-quasi-orders (see, e.g., [3] and [4]). The second part applies the main result of Part I (Theorem III.1) to settle the decidability status of several problems concerning the "similarity of derivations" in different EOL systems (forms). In particular we prove that the "one-to-many simulation" among EOL forms is decidable, settling in this way an open problem from [5]. This result together with [1] says that both fundamental simulation lemmas for EOL forms are effective.

In this section we recall some basic notions concerning EOL systems and establish the notation used in our paper.

(i) For a set X, # X denotes its cardinality. For a finite set of integers X, max X (min X respectively) denotes the greatest element (smallest element respectively) of X. We often identify a singleton $\{x\}$ with its element x and then write x rather than $\{x\}$. We also use $\mathbb{N}$ and $\mathbb{N}^+$ to denote the sets of nonnegative and positive integers respectively.

An alphabet is a finite nonempty set of symbols.

(ii) We use $\wedge$ and $\vee$ to denote the conjunction and the disjunction operators respectively.

(iii) Λ denotes the empty word; given a word x, $|x|$ denotes its length, alph x denotes the set of all letters occurring in x and, for an alphabet Δ, $\#_\Delta x$ denotes the number of occurrences of letters from Δ in x. For a language K, alph $K = \bigcup_{x \in K}$ alph x.

Let Σ be an alphabet and $x,y \in \Sigma^*$; we say that x is a sparse subword of y and we denote $x \prec y$ if $x = a_1 a_2 \ldots a_n$, $n \geq 0$, $a_i \in \Sigma$ for $1 \leq i \leq n$ and $y = u_0 a_1 u_1 a_2 u_2 \ldots a_n u_n$, $u_i \in \Sigma^*$ for $0 \leq i \leq n$. We say that x is a subword of y if $y = uxv$ for $u,v \in \Sigma^*$. For a word x, sub x denotes the set of all subwords of x.

(iv) Let K_1, K_2 be languages. K_1 and K_2 are considered equal if $K_1 \cup \{\Lambda\} = K_2 \cup \{\Lambda\}$.

(v) With each positive integer n, a symbol $\overline{n}$ is associated. Then if x is a word we often consider the word $\overline{x} = \overline{1}\,\overline{2} \ldots \overline{|x|}$. In the case $x = \Lambda$, $\overline{x} = \overline{1}\,\overline{2} \ldots \overline{|x|} = \Lambda$.

(vi) An EOL system is a four-tuple $G = (\Sigma, \varphi, \omega, \Delta)$ where Σ is the total alphabet, $\Delta \subseteq \Sigma$ is the terminal alphabet, $\omega \in \Sigma^+$ is the axiom and φ is a finite substitution on Σ (into the set of subsets of Σ^*). The elements of $\Sigma \setminus \Delta$ are called nonterminals.

The language of G, denoted L(G), is defined by $L(G) = (\bigcup_{i\geq 0} \varphi^i(\omega)) \cap \Delta^*$.

If $x \in \varphi(\alpha)$ for $\alpha \in \Sigma$, then we often write $\alpha \to x$ and we also say that $\alpha \to x$ is a production of G (we write $\alpha \underset{\varphi}{\to} x$ as an abbreviation of " $\alpha \to x$ is a production of G ").

If $\pi = \alpha \to x$ is a production of G, we use lhs π to denote the left-hand side of π (thus lhs $\pi = \alpha$) and rhs π to denote the right-hand side of π (thus rhs $\pi = x$).

Let $x \in \Sigma^*$. If $y \in \varphi(x)$ we often write $x \underset{G}{\Rightarrow} y$ or $x \underset{\varphi}{\Rightarrow} y$ (note that in this way $\Lambda \underset{\varphi}{\Rightarrow} \Lambda$). We also write $x \overset{+}{\underset{G}{\Rightarrow}} y$ and $x \overset{+}{\underset{\varphi}{\Rightarrow}} y$ ($x \overset{*}{\underset{G}{\Rightarrow}} y$ and $x \overset{*}{\underset{\varphi}{\Rightarrow}} y$ respectively) if there exists an $n > 0$ ($n \geq 0$ respectively) such that $y \in \varphi^n(x)$.

G is called reduced if for each $\alpha \in \Sigma$, $\omega \overset{*}{\underset{G}{\Rightarrow}} u\alpha v$ for some $u,v \in \Sigma^*$.

Finally if for each $\alpha \in \Sigma$, $A_\alpha = \{x : x \in \varphi(\alpha)\}$, then

det φ = max $\{\# A_\alpha : \alpha \in \Sigma\}$.

(vii) Let $G = (\Sigma,\varphi,\omega,\Delta)$ be an EOL system and let ℓ be a positive integer. A derivation in G of length ℓ leading from $x \in \Sigma^*$ to $y \in \Sigma^*$ is a sequence $(x = x_0, x_1, \ldots, x_\ell = y)$, such that $x_0 \underset{G}{\Rightarrow} x_1$, $x_1 \underset{G}{\Rightarrow} x_2, \ldots, x_{\ell-1} \underset{G}{\Rightarrow} x_\ell$ together with a precise description of how all the occurrences in x_i are rewritten to obtain x_{i+1} for $0 \leq i \leq \ell-1$. Such a description can be formalized (see, e.g., [6]). We depict such a derivation D by

$$D : x_0 \underset{G}{\Rightarrow} x_1 \underset{G}{\Rightarrow} x_2 \underset{G}{\Rightarrow} \quad \ldots \quad \underset{G}{\Rightarrow} x_\ell \text{ , or by}$$

$$D : x_0 \underset{\varphi}{\Rightarrow} x_1 \underset{\varphi}{\Rightarrow} x_2 \underset{\varphi}{\Rightarrow} \quad \ldots \quad \underset{\varphi}{\Rightarrow} x_\ell .$$

We say that D is a derivation in G if $x_0 = \omega$. The sequence of words $(x_0, x_1, \ldots, x_\ell)$ is called the trace of D, denoted as trace D and the sequence of words $(x_1, \ldots, x_{\ell-1})$ is called the intermediate trace of D, denoted as itrace D.

If we deal with a finite substitution φ on an alphabet θ (into the set of subsets of θ^*), $x_0 \in \theta^*$ and a positive integer ℓ, then we refer to

$$D : x_0 \underset{\varphi}{\Rightarrow} x_1 \underset{\varphi}{\Rightarrow} x_2 \underset{\varphi}{\Rightarrow} \quad \ldots . \quad \underset{\varphi}{\Rightarrow} \; x_\ell$$

as a derivation in the sense that we consider φ to be the finite substitution associated with an EOL system.

(viii) Let $G = (\Sigma,\varphi,\omega,\Delta)$ be a reduced EOL system and let $\alpha \in \Sigma$. Then the <u>set of surroundings of</u> α in G, denoted $\underline{sur}_G\ \alpha$ (or $\underline{sur}\ \alpha$ if G is understood) is defined as follows (see [1]).

$\underline{sur}_G\ \alpha = \{Z_1,Z_2,\ldots,Z_k\}$ where $k \geq 1$, $Z_i \subseteq \Sigma$ for $1 \leq i \leq k$, and

(1) $Z_i \nsubseteq Z_j$, for all $i,j \in \{1,\ldots,k\}$, $i \neq j$,

(2) for all $x,y \in \Sigma^*$, if $\omega \overset{*}{\underset{G}{\Rightarrow}} x\alpha y$, then there exists an i, $1 \leq i \leq k$, such that $Z_i \subseteq \underline{alph}\ xy$,

(3) for each i, $1 \leq i \leq k$, there exist $x,y \in \Sigma^*$ such that $\omega \overset{*}{\underset{G}{\Rightarrow}} x\alpha y$ and $(\underline{alph}\ xy)\setminus\{\alpha\} = Z_i$.

We recall the following lemma from [1].

<u>Lemma I.1</u>. Let $G = (\Sigma,\varphi,\omega,\Delta)$ be a reduced EOL system and let $\alpha \in \Sigma$. Then $\underline{sur}_G\ \alpha$ is effectively computable. □

For unexplained notions and terminology concerning EOL systems we refer to [6].

(ix) We conclude this section by recalling a result from [3] which will be quite useful in the rest of our paper.

<u>Theorem I.1</u>. Let Σ be an alphabet and let $K \subseteq \Sigma^*$. There exists a finite subset B of K (possibly empty) such that for each word $w \in K$, there exist a word $v \in B$ such that $v \prec w$. □

If K and B are as in the above theorem and moreover $K = \{w :$ there exists a $v \in B$ such that $\underline{alph}\ w = \underline{alph}\ v$ and $v \prec w\}$, then B is called a <u>base for</u> K. A base B for K is called <u>minimal</u> if no $B_1 \subsetneq B$ exists such that B_1 is a base for K.

II. BASIC FORMULAS AND LANGUAGES

Throughout the paper we assume Σ_u to be a fixed infinite set of symbols; all considered alphabets will be finite nonempty subsets of Σ_u. Furthermore ξ denotes a variable which ranges over Σ_u^*. We need the following two predicates. Let $b \in \Sigma_u$, $w \in \Sigma_u^*$ and let n be a nonnegative integer. Then

$\underline{me}(b,n,w)$ if and only if $\#_b\, w \geq n$, and

$\underline{ez}(b,w)$ if and only if $\#_b\, w = 0$.

Note that "me" and "ze" abbreviate phrases "more than or equal to" and "equal to zero" respectively.

Definition. The set of basic formulas, denoted F is defined inductively as follows.

(1) For each $b \in \Sigma_u$ and each nonnegative integer n, $\underline{me}(b,n,\xi) \in F$ and $\underline{ez}(b,\xi) \in F$ (those formulas are referred to as atomic formulas).

(2) If $\Phi(\xi) \in F$ and $\Psi(\xi) \in F$ then $\Phi(\xi) \wedge \Psi(\xi) \in F$ and $\Phi(\xi) \vee \Psi(\xi) \in F$.

(3) No other formulas belong to F. □

According to the above definition a basic formula $\Phi(\xi)$ is built up from a finite number of atomic formulas using the operations $\wedge$ and $\vee$. Atomic formulas occurring in $\Phi(\xi)$ will be referred to as components of $\Phi(\xi)$. Components of the form $\underline{me}(b,n,\xi)$ are referred to as positive components of $\Phi(\xi)$ and components of the form $\underline{ez}(b,\xi)$ are referred to as negative components of $\Phi(\xi)$.

Definition. Let $\Phi(\xi) \in F$. Then

$\underline{alph}\ \Phi(\xi) = \{b \in \Sigma_u :$ b equals the first argument of a component of $\Phi(\xi)\}$. □

We also consider the following subsets of the set of basic formulas.

Definition. Let $\Phi(\xi) \in F$ and let Ω be an alphabet. Then $\Phi(\xi)$ is called Ω-positive if

(1) $\Omega \subseteq$ alph $\Phi(\xi)$, and

(2) no element of Ω appears as the first argument in a negative component of $\Phi(\xi)$. □

A basic formula $\Phi(\xi)$ defines in a natural way a "basic language", that is the set of all words in alph $\Phi(\xi)$ that satisfy $\Phi(\xi)$.

Definition. Let $\Phi(\xi) \in F$.

Then define $L(\Phi(\xi)) = \{w \in (\text{alph}\ \Phi(\xi))^* : \Phi(w)\}$.

A language K is said to be a basic language if $K = L(\Phi(\xi))$ for some basic formula $\Phi(\xi)$.

For an alphabet Ω we say that a language K is Ω-positive if $K = L(\Phi(\xi))$ for some Ω-positive formula $\Phi(\xi)$. □

We are going now to develop a number of "normal form" results concerning basic formulas and basic languages. First we need the following definition.

Definition. Let $\Phi(\xi) \in F$.

Then $\Phi(\xi)$ is said to be in disjunctive normal form if

$\Phi(\xi) = \Phi_1(\xi) \vee \Phi_2(\xi) \vee \ldots \vee \Phi_k(\xi)$ for some $k \geq 1$, and for $1 \leq i \leq k$,

$\Phi_i(\xi) = \Phi_{i,1}(\xi) \wedge \Phi_{i,2}(\xi) \wedge \ldots \wedge \Phi_{i,p_i}(\xi)$ where $p_i \geq 1$ and for $1 \leq j \leq p_i$,

$\Phi_{i,j}(\xi)$ is an atomic formula.

In the above each $\Phi_i(\xi)$, $1 \leq i \leq k$, is called a disjunct (of $\Phi(\xi)$).

For $1 \leq i \leq k$, $1 \leq j \leq p_i$, each $\Phi_{i,j}(\xi)$ is called a conjunct (of $\Phi_i(\xi)$). □

When we restrict our attention to basic formulas over a fixed alphabet, we also need the following definitions.

Definition. Let Σ be an alphabet.

(1) F_Σ, the set of all basic formulas over Σ is defined by

$F_\Sigma = \{\Phi(\xi) \in F : \text{alph}\ \Phi(\xi) = \Sigma\}$.

(2) Let $\Phi_1(\xi)$ and $\Phi_2(\xi)$ be two elements of F_Σ. We say that $\Phi_1(\xi)$ implies $\Phi_2(\xi)$ if and only if $L(\Phi_1(\xi)) \subseteq L(\Phi_2(\xi))$. $\Phi_1(\xi)$ and $\Phi_2(\xi)$ are called equivalent if and only if $L(\Phi_1(\xi)) = L(\Phi_2(\xi))$.

(3) The relation $\lessdot$ on elements of F_Σ is defined as follows. Let $\Phi_1(\xi)$, $\Phi_2(\xi) \in F_\Sigma$. Then $\Phi_1(\xi) \lessdot \Phi_2(\xi)$ if and only if $\Phi_2(\xi)$ implies $\Phi_1(\xi)$.

(4) $\Phi(\xi) \in F_\Sigma$ is said to be in strong disjunctive normal form if $\Phi(\xi)$ is in disjunctive normal form and for each disjunct $\Psi(\xi)$ of $\Phi(\xi)$ there exist $\ell,m \geq 0$ such that

$\Psi(\xi) = \underline{ez}(b_1,\xi) \wedge \underline{ez}(b_2,\xi) \wedge \ldots \wedge \underline{ez}(b_\ell,\xi) \wedge \underline{me}(\overline{b}_1,n_1,\xi) \wedge \underline{me}(\overline{b}_2,n_2,\xi)$
$\wedge \ldots \wedge \underline{me}(\overline{b}_m,n_m,\xi)$

where each letter of Σ occurs in precisely one conjunct of $\Psi(\xi)$ and

$n_1 \leq n_2 \leq \ldots \leq n_m$.

If $\Phi(\xi)$ is in strong disjunctive normal form and $\Psi(\xi)$ is a disjunct of $\Phi(\xi)$ as above then

$\underline{type}\ \Psi(\xi) = (\{b_1,b_2,\ldots,b_\ell\}, (\overline{b}_1,\overline{b}_2,\ldots,\overline{b}_m))$ is called the type of $\Psi(\xi)$ (over Σ). If $\ell = 0$ ($m = 0$ respectively) then the first (respectively second component of $\underline{type}\ \Psi(\xi)$ is set to be the empty set. □

The following result indicates the usefulness of the disjunctive and the strong disjunctive normal form.

Lemma II.1. Let Σ,Ω be alphabets and let $\Phi(\xi) \in F_\Sigma$. Then one can effectively construct an equivalent $\Gamma(\xi) \in F_\Sigma$ in disjunctive normal form. Moreover

(1) if $L(\Phi(\xi)) \neq \emptyset$, then $\Gamma(\xi)$ is in strong disjunctive normal form, and

(2) $\Gamma(\xi)$ is Ω-positive if $\Phi(\xi)$ is Ω-positive.

Proof.

Let $\Phi(\xi)$ be as in the statement of the lemma. Obviously we can apply in $\Phi(\xi)$ the distributive laws for $\wedge$ and $\vee$ to get effectively an equivalent $\Phi'(\xi) \in F_\Sigma$ in disjunctive normal form, Ω-positive in the case $\Phi(\xi)$ is Ω-positive.

Moreover if $L(\Phi(\xi)) \neq \emptyset$, for each disjunct $\Psi(\xi)$ of $\Phi'(\xi)$ such that $\{x \in \Sigma^* : \Psi(x)\} \neq \emptyset$ and each $b \in \Sigma$, let

(i) $\Psi_b(\xi) = \underline{ez}(b,\xi)$, if $\underline{ez}(b,\xi)$ is a component of $\Psi(\xi)$,

(ii) $\Psi_b(\xi) = \underline{me}(b,n_b,\xi)$, if $b \in \underline{alph}\ \Psi(\xi)$, $\underline{ez}(b,\xi)$ is not a component of $\Psi(\xi)$ and $n_b = \underline{max}\ \{b : \underline{me}(b,n,\xi)$ is a component of $\Psi(\xi)\}$,

(iii) $\Psi_b(\xi) = \underline{me}(b,0,\xi)$ if $b \in \Sigma \setminus \underline{alph}\ \Psi(\xi)$. Then $\overline{\Psi}(\xi) = \bigwedge_{b \in \Sigma} \Psi_b(\xi)$.

Permuting the conjuncts of $\overline{\Psi}(\xi)$ we easily get disjuncts the disjunction of which gives us a basic formula in strong disjunctive normal form, equivalent to $\Phi(\xi)$, Ω-positive if $\Phi(\xi)$ is Ω-positive. □

We now demonstrate that the relation $\lessdot$ is a well-quasi-order on F_Σ, i.e. it is a quasi-order (a reflexive and transitive relation) such that every infinite sequence of elements of F_Σ contains an infinite ascending subsequence. In the proof of the above result we will use a combinatorial result concerning vectors of nonnegative integers, the proof of which is the subject of the Appendix. Precise definitions and terminology concerning well-quasi-orders are also stated there.

Theorem II.1. Let Σ be an alphabet. Then $\lessdot$ is a well-quasi-order on F_Σ.

Proof.

That $\lessdot$ is a quasi-order on F_Σ can be easily verified.

To prove that $\lessdot$ is a well-quasi-order it now suffices to prove that any infinite sequence τ of elements of F_Σ is well-quasi-ordered with respect to $\lessdot$ (see Lemma A.1,(1)).

Therefore assume τ is an infinite sequence of elements of F_Σ. Without loss of generality we can assume that for all $i \geq 1$ such that $L(\tau(i)) \neq \emptyset$, $\tau(i)$ is in strong disjunctive normal form (see Lemma II.1).

For each type π over Σ (observe there is only a finite number of such types) and $\Psi(\xi) \in F_\Sigma$ in strong disjunctive normal form let

$\Psi^{[\pi]}(\xi) = \bigvee \Psi_i(\xi)$

where the disjunction is over all disjuncts $\Psi_i(\xi)$ of $\Psi(\xi)$ such that

$\underline{type}\ \Psi_i(\xi) = \pi$ if there exists a disjunct of $\Psi(\xi)$ of type π,

otherwise

$\Psi^{[\pi]}(\xi)$ = FALSE (FALSE stands for a fixed but arbitrary basic formula over Σ such that its language is empty).

If $\Psi(\xi) \in F_\Sigma$ and $L(\Psi(\xi)) = \emptyset$ then we set $\Psi^{[\pi]}(\xi)$ = FALSE for any type π over Σ.

For every type $\pi = (\{b_1,b_2,\ldots,b_\ell\}, (\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_m))$ over Σ with $m > 0$ we define the sequence $\tau^{[\pi]}$ as follows:

for $i \geq 1$, $\tau^{[\pi]}(i) = (\tau(i))^{[\pi]}$.

Also define the sequence $\rho^{[\pi]}$ as follows:

for $i \geq 1$, $\rho^{[\pi]}(i) = \emptyset$ if $\tau^{[\pi]}(i)$ = FALSE, and

$\rho^{[\pi]}(i) = \{(n_1,n_2,\ldots,n_m)$: a disjunct of $\tau^{[\pi]}(i)$ is of the form $\underline{ez}(b_1,\xi) \wedge \underline{ez}(b_2,\xi) \wedge \ldots \wedge \underline{ez}(b_\ell,\xi) \wedge \underline{me}(\bar{b}_1,n_1,\xi) \wedge \underline{me}(\bar{b}_2,n_2,\xi) \wedge \ldots \wedge \underline{me}(\bar{b}_m,n_m,\xi)\}$.

Then obviously the following claim holds.

Claim II.1. Let $\pi = (\{b_1,b_2, \ldots ,b_\ell\},(\bar{b}_1,\bar{b}_2, \ldots,\bar{b}_m))$ be a type over Σ such that $m > 0$. Then for $1 \leq i_1 < i_2$, $\rho^{[\pi]}(i_1)\ \hat{R}_m\ \rho^{[\pi]}(i_2)$ implies $\tau^{[\pi]}(i_1) \lessdot \tau^{[\pi]}(i_2)$. □

Claim II.1 and Theorem A.1 yield that for every $\pi = (\{b_1,b_2,\ldots,b_\ell\},(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_m))$ over Σ such that $m > 0$, $\tau^{[\pi]}$ is well-quasi-ordered with respect to $\lessdot$.

Clearly also $\tau^{[\Sigma,\emptyset]}$ is well-quasi-ordered with respect to $\lessdot$.

Since for each $i \geq 1$, $\tau(i) = \bigvee_{\pi \text{ type over } \Sigma} \tau^{[\pi]}(i)$, by an application of (2) of Lemma A.1 we get that τ is well-quasi-ordered with respect to $\lessdot$.

This concludes the proof of the theorem. □

Basic languages were defined through basic formulas. It turns out that we can define them in a combinatorial fashion.

Theorem II.2. Let Σ be an alphabet and $K \subseteq \Sigma^*$. Then K is a basic language if and only if

(1) K is permutationally closed, and

(2) for each $w \in K$ and each $u \in \Sigma^*$, if $w \prec u$ and $\underline{alph}\, w = \underline{alph}\, u$ then $u \in K$.

Proof.

The only if-part is obvious.

To prove the if-part, let $K \subseteq \Sigma^*$ such that (1) and (2) hold.

Either $K = \emptyset$ and then $K = L(\underline{me}(b,1,\xi) \wedge \underline{ez}(b,\xi))$ where $b \in \Sigma$, or K can be written as $K = \bigcup_{i=1}^{s} K_i$, $s \geq 1$ where for each $1 \leq i \leq s$, $K_i \neq \emptyset$ and $w_1, w_2 \in K_i$ implies $\underline{alph}\, w_1 = \underline{alph}\, w_2$; moreover, if $1 \leq i < j \leq s$ then $alph\, K_i \neq \underline{alph}\, K_j$.

Then according to Theorem I.1 for each K_i $(1 \leq i \leq s)$ there exists a finite subset B_i of K_i such that for each $w \in K_i$ there is a $u \in B_i$ with $u \prec w$.

Clearly each B_i can be chosen nonempty.

Let $B = \bigcup_{i=1}^{s} B_i$. Assume that $\Sigma = \{a_1, \ldots, a_r\}$. Then for each pair $(i,w) \in \{1,\ldots,r\} \times B$ denote

$$C_{i,w}(\xi) = \begin{cases} \underline{me}(a_i, \#_{a_i} w, \xi) & \text{if } a_i \in \underline{alph}\, w, \\ \underline{ez}(a_i, \xi) & \text{if } a_i \notin \underline{alph}\, w. \end{cases}$$

Finally let $\Phi(\xi) = \bigvee_{w \in B} \left(\bigwedge_{i=1}^{r} C_{i,w}(\xi) \right)$.

Clearly $\Phi(\xi) \in F$ and $L(\Phi(\xi)) = K$. □

Theorem II.3. Let K be a basic language and let Ω be an alphabet. Then K is an Ω-positive language if and only if

(II.1) for each pair $(x,y) \in K \times \Omega^*$, $xy \in K$.

Proof.

The only if-part is obvious.

To prove the if-part let K and Ω be as in the statement of the theorem such that (II.1) holds.

If $K = \emptyset$, let $b \in \Sigma_u \setminus \Omega$ and $\Phi(\xi) = \underline{me}(b,1,\xi) \wedge \underline{ez}(b,\xi) \wedge \bigwedge_{c \in \Omega} \underline{me}(c,0,\xi)$.

Clearly $\Phi(\xi)$ is an Ω-positive formula and $K = L(\Phi(\xi))$.

If $K \neq \emptyset$, assume $K = L(\Phi(\xi))$ where $\Phi(\xi)$ is in disjunctive normal form.

Let $\Phi(\xi) = \Phi_1(\xi) \vee \Phi_2(\xi) \vee \ldots \vee \Phi_k(\xi)$, $k \geq 1$ where for each $1 \leq i \leq k$, $\Phi_i(\xi)$ is a disjunct of $\Phi(\xi)$. Furthermore assume without loss of generality that for $1 \leq i \leq k$ $\underline{alph}\, \Phi_i(\xi) = \underline{alph}\, \Phi(\xi)$ and $L(\Phi_i(\xi)) \neq \emptyset$.

The fact that K is Ω-positive is proved as follows.

Clearly $\Omega \subseteq \underline{alph}\, K \subseteq \underline{alph}\, \Phi(\xi)$ otherwise (since $K \neq \emptyset$) (II.1) cannot be valid.

Then we claim the following.

Claim II.2. Let $\Phi(\xi)$ be as above. If $\underline{ez}(b,\xi)$ occurs as a component of $\Phi(\xi)$ where $b \in \Omega$, then one can construct a basic formula $\Psi(\xi)$ such that

(i) $K = L(\Psi(\xi))$, and

(ii) the number of occurrences of $\underline{ez}(b,\xi)$, $b \in \Omega$, as a component of $\Psi(\xi)$ is smaller than the number of occurrences of $\underline{ez}(b,\xi)$, $b \in \Omega$ as a component of $\Phi(\xi)$.

Proof of Claim II.2.

Let j be an arbitrary but fixed element of $\{1,\ldots,k\}$.

Let $\Phi_j(\xi) = \Phi_{j,1}(\xi) \wedge \Phi_{j,2}(\xi) \wedge \ldots \wedge \Phi_{j,t}(\xi)$, $t \geq 1$ and for $1 \leq i \leq t$, $\Phi_{j,i}(\xi)$ is an atomic formula. Let m be a fixed element of $\{1,\ldots.,t\}$ such that $\Phi_{j,m}(\xi) = \underline{ez}(b,\xi)$ where $b \in \Omega$. Since $L(\Phi_j(\xi)) \neq \emptyset$, (II.1) implies that $k \geq 2$.

Let then $\Phi'(\xi) = \bigvee_{\substack{1 \leq i < j \\ j < i \leq k}} \Phi_i(\xi)$ and if $t > 1$, $\Phi'_j(\xi) = \bigwedge_{\substack{1 \leq i < m \\ m < i \leq t}} \Phi_{j,i}(\xi)$.

(i) If $t > 1$,
$$\begin{aligned} L(\Phi(\xi)) &= L(\Phi'(\xi) \vee \Phi_j(\xi)) = L(\Phi'(\xi) \vee (\underline{ez}(b,\xi) \vee \Phi'_j(\xi))) \\ &= L(\Phi'(\xi) \vee (\underline{ez}(b,\xi) \wedge \Phi'_j(\xi)) \vee (\underline{me}(b,1,\xi) \wedge \Phi'_j(\xi))) \\ &= L(\Phi'(\xi) \vee (\underline{me}(b,0,\xi) \wedge \Phi'_j(\xi))). \end{aligned}$$

Let then $\Psi(\xi) = \Phi'(\xi) \vee (\underline{me}(b,0,\xi) \wedge \Phi'_j(\xi))$.

(ii) If $t = 1$, $L(\Phi(\xi)) = L(\Phi'(\xi) \vee \underline{ez}(b,\xi))$

$$= L(\Phi'(\xi) \vee \underline{ez}(b,\xi) \vee \underline{me}(b,1,\xi))$$

$$= L(\Phi'(\xi) \vee \underline{me}(b,0,\xi)).$$

Let then $\Psi(\xi) = \Phi'(\xi) \vee \underline{me}(b,0,\xi)$.

Then clearly Claim II.2 holds. □

Iterating the above construction a finite number of times, we end with an Ω-positive formula the language of which equals K. Hence the theorem holds. □

III. THE T-OPERATOR

Throughout this section θ will always denote an alphabet and Ω will denote either $\{\Lambda\}$ or an alphabet such that $\theta \cap \Omega = \emptyset$. Further φ and ψ denote finite substitutions on θ and Ω respectively. We also need the following definition.

Definition. Let $w \in K \subseteq \theta^*\Omega^*$, $w = w_1w_2$ where $w_1 \in \theta^*$ and $w_2 \in \Omega^*$. Then w_1 is called the θ-part of w and w_2 is called the Ω-part of w. □

The following operator on languages will be a basic tool in the rest of the paper.

Definition. Let $K \subseteq \theta^*\Omega^*$.
Then $T_{\theta,\Omega}(\varphi,\psi,K) = \{\alpha\beta : \alpha \in \theta^*, \beta \in \Omega^*$, there exists an $x \in \varphi(\alpha)$ such that $\{x\}\ \psi(\beta) \subseteq K\}$. □

Whenever θ and Ω are understood we write $T(\varphi,\psi,K)$ rather than $T_{\theta,\Omega}(\varphi,\psi,K)$.

Observe that in the case $\Omega = \{\Lambda\}$, the above definition (recall our convention that $\Lambda \underset{\psi}{\Rightarrow} \Lambda$ for any finite substitution ψ) reduces to
$T(\varphi,\psi,K) = \{\alpha \in \theta^*$: there exists an $x \in \varphi(\alpha)$ such that $x \in K\} = \varphi^{-1}(K)$.

The following lemma deals with the behaviour of basic languages under the T-operator.

Lemma III.1. Let $K = \theta^*\Omega^* \cap \overline{K}$ where $\overline{K} = L(\Phi(\xi))$, $\Phi(\xi) \in F$ and $\Phi(\xi)$ is Ω-positive if Ω is an alphabet. Then one can effecfively construct a basic language $\overline{K}'$ such that $T(\varphi,\psi,K) = \theta^*\Omega^* \cap \overline{K}'$ where $\overline{K}'$ is Ω-positive if Ω is an alphabet.

Proof.

(1) Let $K' = T(\varphi,\psi,K)$.

First we prove that K' is of the desired form.

Clearly if $K' = \emptyset$ then the result holds. Therefore assume $K' \neq \emptyset$.

Obviously $K' \subseteq \theta^*\Omega^*$. Then using Theorems II.2 and II.3, it suffices to prove that K' satisfies the following conditions.

(i) K' is closed with respect to permutations on the θ-part.

(ii) K' is closed with respect to permutations on the Ω-part.

(iii) K' is closed under subword extension on the θ-part, i.e., if $w = \alpha\beta \in K'$, $\alpha \in \theta^*$, $\beta \in \Omega^*$ and $\alpha \prec \alpha'$ such that $\underline{\text{alph}}\ \alpha = \underline{\text{alph}}\ \alpha'$, then $\alpha'\beta \in K'$.

(iv) If $w = \alpha\beta \in K'$, $\alpha \in \theta^*$, $\beta \in \Omega^*$ and $\beta \prec \beta'$ such that $\beta' \in \Omega^*$, then $\alpha\beta' \in K'$.

Conditions (i) and (ii).

Let $\gamma = \alpha\beta \in K'$, $\alpha \in \theta^*$, $\beta \in \Omega^*$, let α' be a permutation of α and β' a permutation of β.

We have to prove that $\alpha'\beta' \in K'$.

Since $\alpha\beta \in K'$, there exists an $x \in \varphi(\alpha)$ such that for all $y \in \psi(\beta)$, $xy \in K$.

Assume $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, $n \geq 0$, $\alpha_i \in \theta$ for $1 \leq i \leq n$. Then α' can be written as $\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_n}$ and $x = x_1x_2\ldots x_n$ where for $1 \leq i \leq n$, $x_i \in \varphi(\alpha_i)$.

Let then $x' = x_{i_1}x_{i_2}\ldots x_{i_n}$. Clearly $x' \in \varphi(\alpha')$ and x' results from x by permuting its letters. Let z' be an arbitrary element of $\psi(\beta')$; clearly there exists a $z \in \psi(\beta)$ such that z' results from z by a permutation of its letters. Then since $xz \in K$, the fact that $K = \theta^*\Omega^* \cap \bar{K}$ with $\bar{K}$ a basic language implies $x'z' \in K$. Since $x' \in \varphi(x')$ and z' was an arbitrary element of $\psi(\beta')$, $\alpha'\beta' \in K'$.

Conditions of (iii) and (iv).

Let $\gamma = \alpha\beta \in K'$, $\alpha \in \theta^*$, $\beta \in \Omega^*$, let $\alpha \prec \alpha'$ and $\beta \prec \beta'$ such that $\underline{\text{alph}}\ \alpha = \underline{\text{alph}}\ \alpha'$ and $\beta' \in \Omega^*$.

Then we have to prove that $\alpha'\beta' \in K'$.

Assume $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, $n \geq 0$, $\alpha_i \in \theta$ for $1 \leq i \leq n$; $\alpha' = u_0\alpha_1u_1\alpha_2u_2\ldots\alpha_nu_n$, $u_i \in (\underline{\text{alph}}\ \alpha)^*$ for $0 \leq i \leq n$; $\beta = \beta_1\beta_2\ldots\beta_m$, $m \geq 0$, $\beta_i \in \Omega$ for $1 \leq i \leq m$; $\beta' = v_0\beta_1v_1\beta_2v_2\ldots\beta_mv_m$, $v_i \in \Omega^*$ for $0 \leq i \leq m$.

Further we know the existence of an $x \in \varphi(\alpha)$ such that for all $y \in \psi(\beta)$, $xy \in K$.

Let $x = x_1 x_2 \ldots x_n$, $x_i \in \varphi(\alpha_i)$ for $1 \le i \le n$. Define φ' homomorphism or $(\underline{\text{alph}}\ \alpha)^*$ as follows: for $1 \le i \le n$, $\varphi'(\alpha_i) = \{x_j$: j is the smallest i, $1 \le i \le n$ such that $\alpha_j = \alpha_i\}$.

Then let $x' = \varphi'(u_0)\ x_1\ \varphi'(u_1)\ x_2\ \varphi'(u_2)\ \ldots\ x_n\ \varphi'(u_n)$; obviously $x' \in \varphi(\alpha')$.
The fact that $\alpha'\beta' \in K'$ is now proved by establishing the following claim.

Claim III.1. Let $\alpha, \beta, \alpha', \beta'$, x and x' be as above.
If there exists a $z' \in \psi(\beta')$ such that $x'z' \notin K$, then there exists a $z \in \psi(\beta)$ such that $xz \notin K$.

Proof of Claim III.1.

Let $z' = v_0'\ w_1 v_1' w_2 v_2' \ldots w_m v_m'$ where for $0 \le i \le m$, $v_i' \in \psi(v_i)$ and for $1 \le i \le m$, $w_i \in \psi(\beta_i)$. Then define $z = w_1 w_2 \ldots w_m$. Clearly $z \in \psi(\beta)$.
We now prove that for this particular z the claim holds.
Let $\overline{K} = L(\Phi(\xi))$. Without loss of generality we can assume $\Phi(\xi)$ to be in disjunctive normal form, $\Phi(\xi) = \Phi_1(\xi) \vee \Phi_2(\xi) \vee \ldots \vee \Phi_k(\xi)$, $k \ge 1$ and for $1 \le i \le k$, $\Phi_i(\xi)$ is a disjunct of $\Phi(\xi)$. Moreover we can assume that $\underline{\text{alph}}\ \Phi(\xi) = \theta$ in the case $\Omega = \{\Lambda\}$, and $\underline{\text{alph}}\ \Phi(\xi) = \theta \cup \Omega$ and $\Phi(\xi)$ Ω-positive in the case Ω equals an alphabet.
Let $\Phi_i(\xi)$ be an arbitrary disjunct of $\Phi(\xi)$ $(1 \le i \le k)$.
Since $x'z' \notin K$, $\Phi_i(x'z')$ must be false. $\Phi_i(\xi)$ consists of a finite number of positive and negative components.
Observe that

(a) for every $b \in \theta$ and every nonnegative integer n,

if $\underline{\text{me}}(b,n,x)$ then $\underline{\text{me}}(b,n,x')$, and

if $\underline{\text{ez}}(b,x)$ then $\underline{\text{ez}}(b,x')$,

(b) in case Ω is an alphabet, for every $b \in \Omega$ and every positive integer n,

if $\underline{\text{me}}(b,n,z)$ then $\underline{\text{me}}(b,n,z')$.

The above observations together with the form of $\Phi_i(\xi)$ immediately yield: if $\Phi_i(xz)$ holds then $\Phi_i(x'z')$ holds. Thus conversely if $\Phi_i(x'z')$ does not hold then $\Phi_i(xz)$ cannot be valid. Since $\Phi_i(x'z')$ is false for every $1 \le i \le k$ and the above reasoning was independent of i we have: $x'z' \notin K$ implies

$x_2 \notin K$ wich proves Claim III.1. □

This concludes the proof of the first part of the theorem.

(2) In this part we prove that $\overline{K'}$ can be constructed effectively. To this aim we will construct a base B_1 for $\overline{K'}$. Then applying the proof of Theorems II.2 and II.3 a basic formula for $\overline{K'}$ can be found effectively. To construct B_1 we first prove effectively a bound p, such that $\theta^* \Omega^* \cap \overline{K'}$ has a base $B \subseteq \theta^* \Omega^*$ the elements of which are all of length smaller or equal to p. To determine B check for each element x of $\theta^* \Omega^*$ of length smaller or equal to p whether or not x belongs to $T(\varphi,\psi,K)$. Clearly, since $K = \theta^* \Omega^* \cap \overline{K}$ and $\overline{K}$ is a basic language given by the basic formula $\Phi(\xi)$, this can be done effectively. Finally let $B = \{x \in T(\varphi,\psi,K) : |x| \leq p\}$ and B_1 = ~~{y : there is an x~~ ∈ B such that y results from x by a permutation of its letters}.

Let n_{max} = max{n : $n \in \mathbb{N}^+$ and n occurs as a second argument in a component of $\Phi(\xi)$} if $\Phi(\xi)$ contains at least one positive component with a positive integer as a second component, and $n_{max} = 1$ otherwise.

Let $p = n_{max} \cdot \underline{max}\{\underline{det}\ \varphi, \underline{det}\ \psi\} . \#(\theta \cup \Omega)$. The fact that it suffices to consider words of $\theta^*\Omega^*$ of length not greater than p is proved by the following claim.

Claim III.2. Let p be as above. Let $\gamma = \alpha\beta \in K'$, $\alpha \in \theta^*$, $\beta \in \Omega^*$ such that $|\gamma| >$ p. Then there exists a $\gamma' \in K'$ such that $\underline{alph}\ \gamma = \underline{alph}\ \gamma'$, $\gamma' \neq \gamma$ and $\gamma' \prec \gamma$.

Proof of Claim III.2.

Let γ be as in the statement of the claim. We have to consider two cases.

(2.i) There exists a $b \in \theta$ such that b occurs more than $n_{max} \cdot \underline{det}\ \varphi$ times in α.

Since $\alpha\beta \in K'$ there must be an $x \in \varphi(\alpha)$ such that $x\,\psi(\beta) \subseteq K$. At least $t > n_{max}$ occurrences of b must be substituted by the same element of $z \in \varphi(b)$. Let α' result from x by removing one such occurrence of b and let x' result from x by removing the corresponding part $\varphi(b)$. Observe that $x' \in \varphi(\alpha')$, $\underline{alph}\ x' = \underline{alph}\ x$ and for every positive component $\underline{me}(c,n,\xi)$ of $\Phi(\xi)$ with $c \in \theta$, if $\underline{me}(c,n,x)$ then $\underline{me}(c,n,x')$. (If $c \notin \underline{alph}\ z$ this is trivial, if $c \in \underline{alph}\ z$ then $\#_c\ x > n_{max}$ and thus $\#_c\ x' \geq n_{max}$.)

Thus $x' \psi(\beta) \subseteq K$ and hence if we set $\gamma' = \alpha'\beta$ we are done.

(2.ii) There exists a $b \in \Omega$ such that b occurs more than $n_{max} \cdot \underline{\det} \psi$ times in β.

Let β' result form β by removing one occurrence of b. Let x be as in (2.i). Let $y' \in \psi(\beta')$. Then we prove that $xy' \in K$. Clearly for all $z \in \psi(b)$, $xy'z \in K$. At least $t \geq n_{max}$ occurrences of b in β are replaced by the same $\overline{z} \in \psi(b)$ to get y'. Choose $z = \overline{z}$. Then analogously to (2.i) we can prove $xy' \in K$ and thus γ' can be taken equal to $\alpha\beta'$. □

This concludes the second part of the proof of the theorem. □

The T-operator now will be used to define two sequences of languages as follows.

Let $K_1 = \theta^*\Omega^* \cap \overline{K}_1$ and $K_2 = \theta^*\Omega^* \cap \overline{K}_2$ where $\overline{K}_1$ and $\overline{K}_2$ are basic languages which are Ω-positive if Ω is an alphabet. Then we define infinite sequences of languages.

$\tau(K_1,K_2) = M_0,M_1,M_2, \ldots$, and

$\rho(K_1,K_2) = L_1,L_2, \ldots$,

as follows:

$M_0 = K_1$, and, for $i > 0$, $L_i = T(\varphi,\psi,M_{i-1})$ and $M_i = L_i \cap K_2$.

In the rest of this section we will assume $K_1,K_2,\tau(K_1,K_2) = M_0,M_1,M_2, \ldots$, and $\rho(K_1,K_2) = L_1,L_2, \ldots$ to be as above. Figure 1 depicts the sequences $\tau(K_1,K_2)$ and $\rho(K_1,K_2)$. We also depict the situation in the case $\Omega = \{\Lambda\}$ (see figure 2).

Using the above definitions, for every positive integer k, L_k can be characterized as follows.

<u>Lemma III.2</u>. Let k be a positive integer. Then (in the notation as above)

$w \in L_k$

if and only if

(1) $w = \alpha\beta$ with $\alpha \in \theta^*$ and $\beta \in \Omega^*$, and

(2) there exists a derivation $D : \alpha \underset{\varphi}{\Rightarrow} \alpha_1 \underset{\varphi}{\Rightarrow} \alpha_2 \underset{\varphi}{\Rightarrow} \cdots \underset{\varphi}{\Rightarrow} \alpha_k$ such that for all derivations $D' : \beta \underset{\psi}{\Rightarrow} \beta_1 \underset{\psi}{\Rightarrow} \beta_2 \underset{\psi}{\Rightarrow} \cdots \underset{\psi}{\Rightarrow} \beta_k$, $\alpha_k\beta_k \in K_1$ and $\alpha_i\beta_i \in K_2$ for $1 \leq i < k$.

Proof.

The proof goes by induction on k.

If $K = 1$, then

$w \in L_1$

if and only if

$w \in T(\varphi,\psi,K_1)$

if and only if

$w = \alpha\beta$, $\alpha \in \theta^*$, $\beta \in \Omega^*$ and there exists an $x \in \varphi(\alpha)$ such that $x\psi(\beta) \subseteq K_1$. The last conditions can be easily seen to be equivalent with conditons (1) and (2) from the statement of the lemma.

Assume the lemma holds for $1 \leq k \leq t$. Then we prove that the lemma also holds for $k = t + 1$. We have the following.

$w \in L_{t+1}$

if and only if

$w \in T(\varphi,\psi,M_t)$

if and only if

$w \in T(\varphi,\psi,L_t \cap K_2)$

if and only if

(1) holds and there exists an $\alpha_1 \in \varphi(\alpha)$ such that for all $\beta_1 \in \psi(\beta)$, $\alpha_1\beta_1 \in L_t \cap K_2$

if and only if

(1) holds, and

there exists a derivation $D : \alpha \underset{\varphi}{\Rightarrow} \alpha_1$ such that for all $\beta_1 \in \psi(\beta)$ $\alpha_1\beta_1 \in K_2$, and

there exists a derivation $\overline{D} : \alpha_1 \underset{\varphi}{\Rightarrow} \alpha_2 \underset{\varphi}{\Rightarrow} \cdots \underset{\varphi}{\Rightarrow} \alpha_{t+1}$ such that for all $\beta_1 \in \psi(\beta)$ and all derivations $\overline{\overline{D}} : \beta_1 \underset{\psi}{\Rightarrow} \beta_2 \underset{\psi}{\Rightarrow} \cdots \underset{\psi}{\Rightarrow} \beta_{t+1}$, $\alpha_{t+1}\beta_{t+1} \in K_1$ and

$\alpha_i \beta_i \in K_2$ for $2 \leq i < t+1$

if and only if

(1) and (2) hold with $k = t+1$. □

We are ready now to state the fundamental decidability result concerning the sequence $\rho(K_1, K_2)$. The result will be a basic tool in the applications of the theory of basic formulas to EOL systems and forms (see [5]).

Theorem III.1. For each $x \in \theta^* \Omega^*$ it is decidable whether or not there exists a positive integer k such that $x \in L_k$.

Proof.

By Lemma III.1, its proof and the obvious fact that the intersection of two basic (respectively Ω-positive) languages is again a basic (respectively Ω-posititve) language which can be found effectively, we can effectively one by one generate a sequence of formulas $\Phi_0(\xi)$, $\Phi_1(\xi)$,, $\Phi_2(\xi)$, ... , such that for $i \geq 0$, $\Phi_i(\xi) \in F_{\theta \cup \Omega}$ and $L(\Phi_i(\xi)) \cap \theta^* \Omega^* = M_i$.
Then by Theorem II.1 there exists a positive integer i such that $\Phi_i(\xi)$ implies $\Phi_r(\xi)$ for some $r < i$. Let i_0 be the smallest i such that this happens. Note that i_0 can be effectively computed. Then the theorem is proved by establishing the following claim.

Claim III.3. There exists a positive integer k such that $x \in L_k$

if and only if

$x \in L_1 \cup L_2 \cup \ldots \cup L_{i_0}$.

Proof of Claim III.3.

The if-part is trivial.
To prove the only if-part assume $x \in L_k$ for some positive integer k.
Let i_x denote the smallest positive integer i such that $x \in L_{i_x}$. The fact that $i_x \leq i_0$ is proved by contradiction as follows. Assume $i_x > i_0$.
Then Lemma III.2 implies

(1) $x = \alpha\beta$, $\alpha \in \theta^*$, $\beta \in \Omega^*$ and

(2) there exists a derivation $D : \alpha \underset{\varphi}{\Rightarrow} \alpha_1 \underset{\varphi}{\Rightarrow} \alpha_2 \underset{\varphi}{\Rightarrow} \cdots \underset{\varphi}{\Rightarrow} \alpha_{i_x}$ such that for all derivations $D' : \beta \underset{\psi}{\Rightarrow} \beta_1 \underset{\psi}{\Rightarrow} \beta_2 \underset{\psi}{\Rightarrow} \cdots \underset{\psi}{\Rightarrow} \beta_{i_x}$, $\alpha_{i_x}\beta_{i_x} \in K_1$ and $\alpha_i\beta_i \in K_2$ for $1 \leq i < i_x$.

Then there exists a $1 \leq j < i_x$ such that for all $\beta_j \in \psi^j(\beta)$, $\alpha_j\beta_j \in L_{i_0} \cap K_2 = M_{i_0}$. Then according to the definition of i_0, there exists an $r < i_0$ such that $\alpha_j\beta_j \in M_r$. Thus $\alpha_j\beta_j \in L_r$. Again applying Lemma III.2, there exists a derivation $\overline{D} : \alpha_j \underset{\varphi}{\Rightarrow} \overline{\alpha}_1 \underset{\varphi}{\Rightarrow} \overline{\alpha}_2 \underset{\varphi}{\Rightarrow} \cdots \underset{\varphi}{\Rightarrow} \overline{\alpha}_r$ such that for all $\beta_j \in \psi^j(\beta)$ and all derivations $\overline{D}' : \beta_j \underset{\psi}{\Rightarrow} \overline{\beta}_1 \underset{\psi}{\Rightarrow} \overline{\beta}_2 \underset{\psi}{\Rightarrow} \cdots \underset{\psi}{\Rightarrow} \overline{\beta}_r$, $\overline{\alpha_r\beta_r} \in K_1$ and $\alpha_i\beta_i \in K_2$ for $1 \leq i < r$. Combining the first j steps of D with $\overline{D}$ and the first j steps of D' with $\overline{D}'$ and again applying Lemma III.2, we get $x \in L_t$ for some $t < i_x$; a contradiction. □

The following result is a useful corollary of Theorem III.1. We need a definition first.

Definition. Let $K, K' \subseteq \theta^*$ be basic languages. Then $g_\theta(\varphi,K,K') = \{x \in \theta^* : x \underset{\varphi}{\Rightarrow} x_1 \underset{\varphi}{\Rightarrow} x_2 \underset{\varphi}{\Rightarrow} \cdots \underset{\varphi}{\Rightarrow} x_\ell$ for some $\ell \geq 1$, $x_\ell \in K'$ and $x_i \in K$ for $1 \leq i < \ell\}$.

Whenever θ is understood we write $g(\varphi,K,K')$ rather than $g_\theta(\varphi,K,K')$.

Corollary III.1. Let $K,K' \subseteq \theta^*$ be basic languages then $g(\varphi,K,K')$ is a basic language which can effectively be computed.

Proof.

Immediately from the proof of Theorem III.1. □

We need the following terminology and notation.

(i) Let Z be a nonempty set and let τ be an infinite sequence of elements of Z. Then for each $i \geq 1$, $\tau(i)$ denotes the i th element of the sequence τ and set $\tau = \{\tau(i) : i \geq 1\}$. The fact that $\bar{\tau}$ is a subsequence of τ is denoted by $\bar{\tau} \ll \tau$.

(ii) Let (Z,R) be a pair where Z is a set and R is a relation on Z. R is called a quasi-order on Z if R is reflexive and transitive. R is called a well-quasi-order on Z if every infinite sequence of elements of Z has an infinite ascending subsequence (see, e.g., [3], [4]) or more precisely stated, for every τ infinite sequence of elements of Z there exists a $\bar{\tau} \ll \tau$, $\bar{\tau}$ infinite, such that for all $i \geq 1$, $\bar{\tau}(i) R \bar{\tau}(i+1)$. Another equivalent formulation (see [3]) is the following: for every infinite sequence τ of elements of Z there exist $1 \leq i < j$ such that $\tau(i) R \tau(j)$. Two elements $b,c \in Z$ are called R-equivalent, denoted $b \equiv_R c$, if bRc and cRb.

(iii) Let (Z,R) be a pair where Z is a set and R is a relation on Z. For each nonnegative integer n, $Z^{(n)}$ denotes the cartesian product of n identical factors Z, i.e. $Z^{(n)} = \underbrace{Z \times Z \times \ldots \times Z}_{n \text{ factors}}$.

Then $R^{(n)}$ is the relation defined on $Z^{(n)}$ as follows. Let $b,c \in Z^{(n)}$, $b = (b_1,b_2,\ldots,b_n)$, $c = (c_1,c_2,\ldots,c_n)$. Then $bR^{(n)}c$ if and only if $b_i R c_i$ for $1 \leq i \leq n$.

Clearly if R is a quasi-order on Z, then $R^{(n)}$ also is a quasi-order on $Z^{(n)}$.

The following definition will be the basic tool in our considerations concerning well-quasi-orders.

Definition. Let (Z,R) be a pair where Z is a set and R is a quasi-order on Z. Let τ be an infinite sequence of elements of Z.

τ is well-quasi-ordered (with respect to R)

if and only if

for every $\bar{\tau} \ll \tau$, $\bar{\tau}$ infinite, there exists a $\bar{\bar{\tau}} \ll \bar{\tau}$, $\bar{\bar{\tau}}$ infinite, such that $\bar{\bar{\tau}}(i) R \bar{\bar{\tau}}(i+1)$ for each $i \geq 1$. □

Then the following result reformulates some results from [3].

Lemma A.1. Let (Z,R) be a pair where Z is a set and R is a quasi-order on Z.

(1) An infinite sequence τ of elements of Z is well-quasi-ordered with respect to R if and only if R is a well-quasi-order on set τ.

(2) Let $\tau_1, \tau_2, \ldots, \tau_n$ be n infinite sequences of elements of Z, $n \geq 1$, well-quasi-ordered with respect to R and let τ be the infinite sequence of elements of $Z^{(n)}$ defined by

$\tau(i) = (\tau_1(i), \tau_2(i), \ldots, \tau_n(i))$ for $i \geq 1$.

Then τ is well-quasi-ordered with respect to $R^{(n)}$.

Proof.

See [3]. □

In the above we have presented a method to construct a sequence τ, well-quasi-ordered with respect to $R^{(n)}$ based on n given sequences well-quasi-ordered with respect to R. Another analogous result will be presented now.

First we need the following definition.

Definition. Let (Z,R) be a pair where Z is a set and R is a relation on Z. Then FIN(Z) = {X : X $\subseteq$ Z and X finite}. (Note that $\emptyset \in$ FIN(Z).)

The relation $\hat{R}$ on FIN(Z) is defined as follows. Let b,c $\in$ FIN(Z). Then

$b\hat{R}c$

if and only if

for every $c' \in c$ there exists a $b' \in b$ such that $b'Rc'$. □

Immediately we get the following results.

Lemma A.2. Let (Z,R) be a pair where Z is a set and R is a quasi-order on Z.

(1) $\hat{R}$ is a quasi-order on FIN(Z).

(2) Let $b \in$ FIN(Z) and $b', b'' \in b$ such that $b'Rb''$ and $b' \not\equiv_R b''$. Then $b \equiv_{\hat{R}} b \setminus \{b''\}$.

Proof.

(1) Obvious

(2) Let b, b', b'' be as in the statement of the lemma. Clearly, since R is reflexive, $b\hat{R}(b \setminus \{b''\})$. Also $(b \setminus \{b''\})\hat{R}\, b$. This is so because for each $c \in b \setminus \{b''\}$, cRc and for b'', $b'Rb''$ where $b' \neq b''$. □

Based on the second part of the above lemma, we define reduced elements of FIN(Z).

Definition. Let Z be a set and R a relation on Z. Let $b \in$ FIN(Z). An element $b'' \in b$ is called redundant if there exists a $b' \in b$ such that $b'Rb''$ and $b' \not\equiv_R b''$. b is called reduced if b contains no redundant elements. □

The above definition allows us to "simplify" elements of FIN(Z) without "loosing any information" concerning $\hat{R}$. Formally we have the following result.

Lemma A.3. Let Z be a set and R a quasi-order on Z. Let τ, μ be infinite sequences of elements of FIN(Z) where for each $i \geq 1$, $\mu(i)$ results from $\tau(i)$ by removing the redundant elements.
Then τ is well-quasi-ordered with repect to $\hat{R}$ if and only if μ is well-quasi-ordered with repect to $\hat{R}$.

Proof.

Obvious. □

Now we are ready to formulate a result analogous to point (2) of Lemma A.1.

Lemma A.4. Let Z be a set and R a quasi-order on Z. Let $\tau_1,\tau_2,\ldots,\tau_n$ be n infinite sequences of elements of FIN(Z), $n \geq 1$, well-quasi-ordered with repect to $\hat{R}$ and let τ be the infinite sequence of elements of FIN(Z) defined by

$\tau(i) = \tau_1(i) \cup \tau_2(i) \cup \ldots \cup \tau_n(i)$ for $i \geq 1$.

Then τ is well-quasi-ordered with repect to $\hat{R}$.

Proof.

Let $\tau,\tau_1,\tau_2,\ldots,\tau_n$ be as in the statement of the lemma. Define the sequence ρ by

$\rho(i) = (\tau_1(i),\tau_2(i),\ldots,\tau_n(i))$ for $i \geq 1$.

Let $\overline{\tau} \ll \tau$, $\overline{\tau}$ infinite, $\overline{\tau}(i) = \tau(j_i)$ for $i \geq 1$.

Let $\overline{\rho}$ be the sequence defined by $\overline{\rho}(i) = \rho(j_i)$ for $i \geq 1$.

Since $\tau_1,\tau_2,\ldots,\tau_n$ are well-quasi-ordered with repect to $\hat{R}$, (2) of Lemma A.1 yields ρ is well-quasi-ordered with repect to $\hat{R}^{(n)}$. Since $\overline{\rho} \ll \rho$, $\overline{\rho}$ infinite, there exists a $\overline{\overline{\rho}} \ll \overline{\rho}$, $\overline{\overline{\rho}}$ infinite, such that $\overline{\overline{\rho}}(i)\ \hat{R}^{(n)}\ \overline{\overline{\rho}}(i+1)$, for $i \geq 1$.

Assume $\overline{\overline{\rho}}(i) = \rho(j_{k_i})$ for $i \geq 1$.

Let then $\overline{\overline{\tau}}(i) = \tau(j_{k_i})$ for $i \geq 1$. Obviously $\overline{\overline{\tau}} \ll \overline{\tau}$ and $\overline{\overline{\tau}}$ is infinite.

Moreover $\overline{\overline{\rho}}(i)\ \hat{R}^{(n)}\ \overline{\overline{\rho}}(i+1)$ for $i \geq 1$,

i.e. $\quad \rho(j_{k_i})\ \hat{R}^{(n)}\ \rho(j_{k_{i+1}})$ for $i \geq 1$,

i.e. $\quad \tau_1(j_{k_i})\ \hat{R}\ \tau_1(j_{k_{i+1}}),\ \tau_2(j_{k_i})\ \hat{R}\ \tau_2(j_{k_{i+1}}),\ldots,\ \tau_n(j_{k_i})\ \hat{R}\ \tau_n(j_{k_{i+1}})$

for $i \geq 1$.

The latter result implies (see definition $\hat{R}$)

$(\tau_1(j_{k_i}) \cup \tau_2(j_{k_i}) \cup \ldots \cup \tau_n(j_{k_i}))\ \hat{R}\ (\tau_1(j_{k_{i+1}}) \cup \tau_2(j_{k_{i+1}}) \cup \ldots \cup \tau_n(j_{k_{i+1}}))$

for $i \geq 1$,

i.e. $\overline{\overline{\tau}}(i)\ \hat{R}\ \overline{\overline{\tau}}(i+1)$ for $i \geq 1$. Thus since $\overline{\tau}$ was an arbitrary infinite sequence such that $\overline{\tau} \ll \tau$, the lemma holds. □

In the rest of this section we will deal with the sets of vectors V_n where n is a positive integer, defined by

$V_n = \{(x_1,x_2,\ldots,x_n) : x_i \in \mathbb{N}$ for $1 \le i \le n$ and $x_i \le x_{i+1}$ for $1 \le i < n\}$.

As relation between elements of $\mathbb{N}$ we use $\le$.

The relation $\le^{(n)}$ on $\mathbb{N}^{(n)}$ will be denoted by R_n.

For $1 \le j \le n$ and $b \in FIN(V_n)$, $\underline{min}_j\, b$ and $\underline{max}_j\, b$ are defined by

$\underline{min}_j\, b = \underline{min}\, \{x_j : (x_1,x_2,\ldots,x_n) \in b\}$, and

$\underline{max}_j\, b = \underline{max}\, \{x_j : (x_1,x_2,\ldots,x_n) \in b\}$.

Now we are ready to formulate and to prove the main theorem of this section.

Theorem A.1. Let n be a positive integer. Then $\hat{R}_n$ is a well-quasi-order on $FIN(V_n)$.

Proof.

Clearly for each positive integer n, $\hat{R}_n$ is a quasi-order on $FIN(V_n)$.
The proof that $\hat{R}_n$ is a well-quasi-order on $FIN(V_n)$ goes by induction on n.
If n = 1 we have to prove that $\hat{R}_1$ is a well-quasi-order on $FIN(V_1)$.
This is proved by contradiction. Assume that $\hat{R}_1$ is not a well-quasi-order on $FIN(V_1)$. Then there exists an infinite sequence τ of elements of $FIN(V_1)$ such that for no $1 \le i < j$, $\tau(i)\hat{R}_1\, \tau(j)$ holds. This means that for all $1 \le i < j$, there exists an $x \in \tau(j)$ such that for every $y \in \tau(i)$, $y > x$. Then obviously τ must be finite; a contradiction. Hence $\hat{R}_1$ is a well-quasi-order on $FIN(V_1)$.
Assume that the theorem holds for $1 \le n \le t$. Then we prove that the theorem also holds for n = t+1.
Let μ denote an infinite sequence of elements of $FIN(V_n)$. If μ contains infinitely many occurrences of the empty set, clearly an infinite ascending subsequence can be found. Otherwise there are two cases to consider.

Case 1. There exists a j, $1 \le j \le n$ such that $\lim_{i\to\infty} (\underline{\min}_j \mu(i)) = \infty$.

Let j be as above. Then $\bar{\mu} \ll \mu$, $\bar{\mu}$ infinite is inductively defined as follows.

$\bar{\mu}(1) = \mu(k)$, where k is the smallest positive integer ℓ such that $\mu(\ell) \neq \emptyset$,

and for $i \ge 1$,

$\bar{\mu}(i+1) = \mu(k)$, where k is the smallest positive integer ℓ such that

$\underline{\min}_j\ \mu(\ell) > \underline{\max}_j\ \bar{\mu}(i)$.

Then construct ν the infinite sequence of elements of FIN(V_{n-1}) where for $i \ge 1$, $\nu(i)$ results from $\bar{\mu}(i)$ by removing the j'th component from all vectors of $\bar{\mu}(i)$. Then we claim the following.

Claim A.1. For $1 \le i_1 < i_2$, $\bar{\mu}(i_1)\ \hat{R}_n\ \bar{\mu}(i_2)$ if and only if $\nu(i_1)\hat{R}_{n-1}\ \nu(i_2)$.

Proof of Claim A.1.

(i) Assume $\bar{\mu}(i_1)\hat{R}_n\ \bar{\mu}(i_2)$ and let $c \in \nu(i_2)$. Then there exists a $\bar{c} \in \bar{\mu}(i_2)$ such that c results from $\bar{c}$ by removing its j'th component. Then we know that there exists a $\bar{b} \in \bar{\mu}(i_1)$ such that $\bar{b}\ R_n\ \bar{c}$. Let b result from $\bar{b}$ by removing its j'th component. Then $b \in \nu(i_1)$ and clearly $b\ R_{n-1}\ c$. Since c was arbitrary, $\nu(i_1)\hat{R}_{n-1}\ \nu(i_2)$ follows.

(ii) Assume $\nu(i_1)\hat{R}_{n-1}\ \nu(i_2)$ and let $\bar{c} \in \bar{\mu}(i_2)$. Let c result from $\bar{c}$ by removing its j'th component. Then there exists a $b \in \nu(i_1)$ such that $b\ R_{n-1}\ c$. Let $\bar{b}$ be an arbitrary element of $\bar{\mu}(i_1)$ such that b results from $\bar{b}$ by removing its j'th component. Since $\underline{\min}_j\ \bar{\mu}(i_2) > \underline{\max}_j\ \bar{\mu}(i_1)$, we have $\bar{b}\ R_n\ \bar{c}$. Since $\bar{c}$ was arbitrary, $\bar{\mu}(i_1)\hat{R}_n\ \bar{\mu}(i_2)$ follows.

This ends the proof of Claim A.1. □

Now from the induction hypothesis follows the existence of $\bar{\nu} \ll \nu$, $\bar{\nu}$ infinite and ascending with repect to $\hat{R}_{n-1}$. Then using Claim A.1 the existence of $\bar{\bar{\mu}} \ll \mu$, $\bar{\bar{\mu}}$ infinite and ascending with repect to $\hat{R}_n$ follows. This ends the proof in Case 1.

Case 2. There exists a nonnegative integer t such that for all j, $1 \le j \le n$ and for each $i \ge 1$, $\underline{\min}_j\ \mu(i) \le t$.

Let t be as above. We prove that μ is well-quasi-ordered with respect to $\hat{R}_n$. Without loss of generality we can assume that all elements of μ are reduced (see Lemma A.3).

Firstly we make the following observation.

(A.1) For $i \geq 1$ and each $x = (x_1,x_2,\ldots,x_n) \in \mu(i)$, there exists an ℓ, $1 \leq \ell \leq k$ such that $x_\ell \leq t$.

The above observation is proved by a contradiction. Let i and x be as stated above. Since $\underline{\min}_n\ \mu(i) \leq t$, there must be an $y = (y_1,y_2,\ldots,y_n) \in \mu(i)$ such that $y_n \leq t$. Assume now that $x = (x_1,x_2,\ldots,x_n) \in \mu(i)$ such that $x_j > t$ for $1 \leq j \leq n$. But then $y\ R_n\ x$ and $x \not R_{R_n}\ y$. This implies that $\mu(i)$ cannot be reduced, a contradiction; hence (A.1) must hold.

Based on μ, now n.t sequences of elements of $FIN(V_n)$ are constructed.

For each pair $(u,v) \in \{1,2,\ldots,n\} \times \{1,2,\ldots,t\}$ and $i \geq 1$ define

$\mu_{u,v}(i) = \{(x_1,x_2,\ldots,x_n) \in \mu(i) : x_u = v\}$.

Obviously

(A.2) for $i \geq 1$, $\mu(i) = \bigcup\limits_{\substack{1 \leq u \leq n \\ 1 \leq v \leq t}} \mu_{u,v}(i)$

We are now going to prove that each $\mu_{u,v}$ is well-quasi-ordered with respect to $\hat{R}_n$. As in Case 1 again an infinite sequence of elements of $FIN(V_{n-1})$ is associated with each $\mu_{u,v}$ as follows.

For $1 \leq u \leq n$, $1 \leq v \leq t$ define the infinite sequence $\nu_{u,v}$ where for $i \geq 1$, $\nu_{u,v}(i)$ results from $\mu_{u,v}(i)$ by removing from each vector its u'th component.

Then we claim the following.

<u>Claim A.2</u>. For $1 \leq i_1 < i_2$, $\mu_{u,v}(i_1)\hat{R}_n\ \mu_{u,v}(i_2)$ if and only if $\nu_{u,v}(i_1)\hat{R}_{n-1}\ \nu_{u,v}(i_2)$.

<u>Proof of Claim A.2</u>.

Analogous to the proof of Claim A.1. □

Since by the inductive assumption $\nu_{u,v}$ is well-quasi-ordered with respect to $\hat{R}_{n-1}$, Claim A.2 implies that $\mu_{u,v}$ is well-quasi-ordered with respect to $\hat{R}_n$

for each pair $(u,v) \in \{1,2,\ldots,n\} \times \{1,2,\ldots,t\}$. Then (A.2) together with Lemma A.4 imply that μ itself is well-quasi-ordered with respect to $\hat{R}_n$ and this implies the existence of an infinite ascending subsequence with respect to $\hat{R}_n$. This ends the proof of Case 2.

Since for every infinite sequence μ of elements of $FIN(V_n)$ we have demonstrated the existence of an infinite ascending subsequence with respect to $\hat{R}_n$, $\hat{R}_n$ is a well-quasi-order on $FIN(V_n)$.

This concludes the proof of the induction step and of the theorem. □

ACKNOWLEDGEMENT

The first and the second author gratefully acknowledge the financial support of NSF grant number MCS 79-03838. The second and the third author are indebted to NFWO Belgium for supporting their research.

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Basic formulas and languages, Part I. The theory

Figure 1

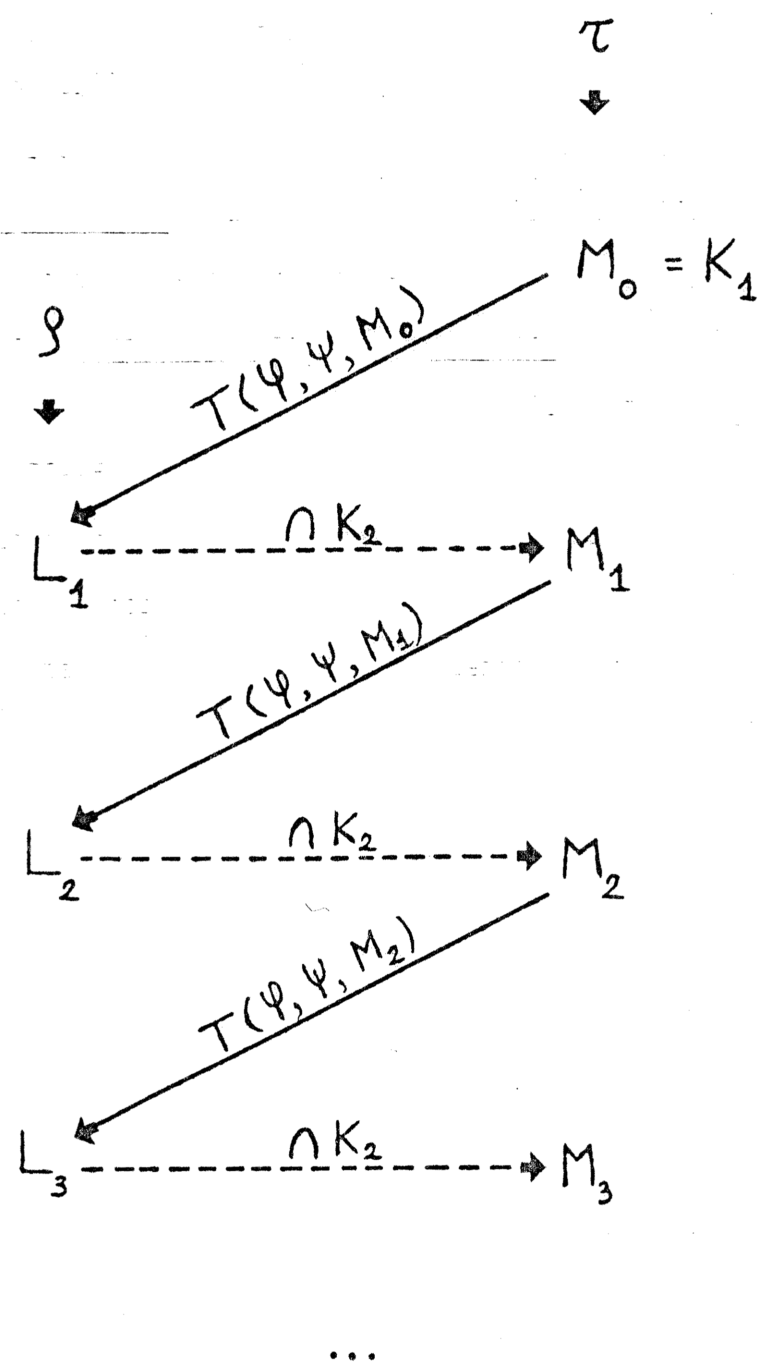
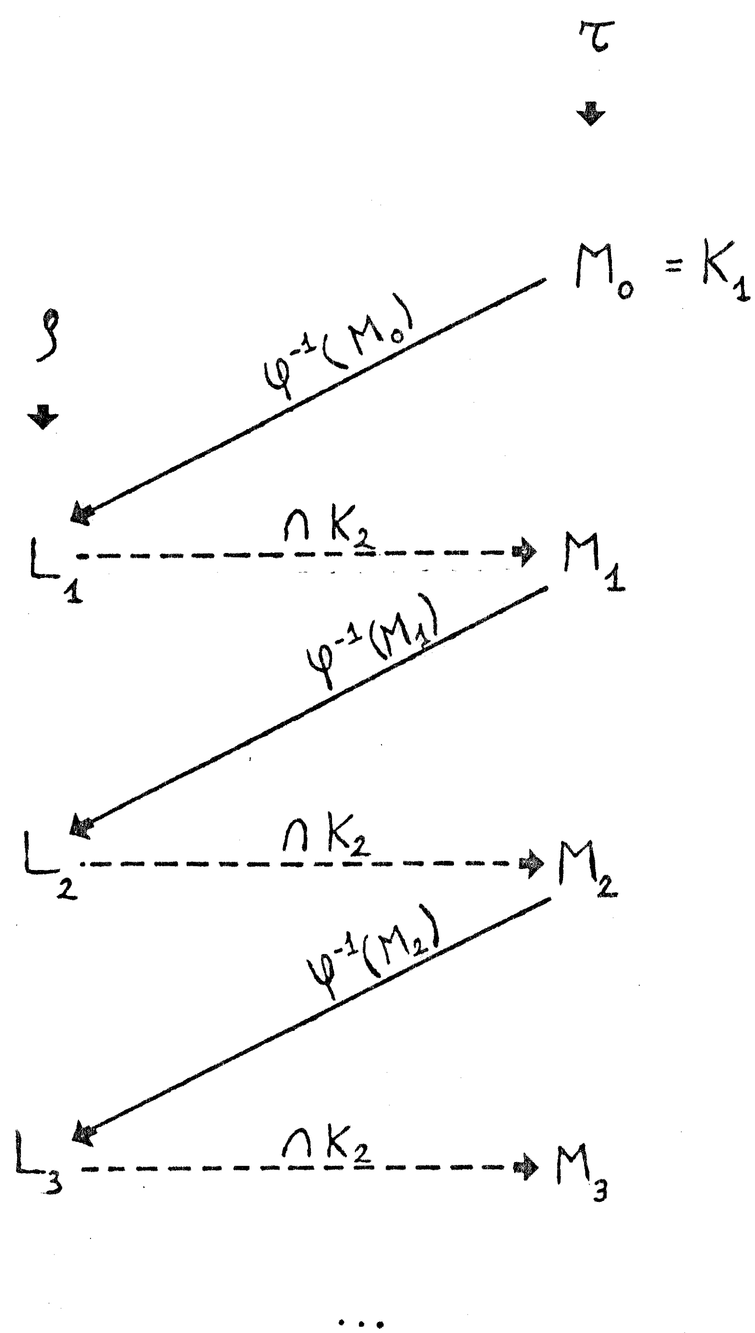


Figure 2


τ
ρ
$M_0 = K_1$
$\varphi^{-1}(M_0)$
L_1
$\cap K_2$
M_1
$\varphi^{-1}(M_1)$
L_2
$\cap K_2$
M_2
$\varphi^{-1}(M_2)$
L_3
$\cap K_2$
M_3
...