Generalized Post Correspondence Problem of Length 2
Part III Decidability

Andrzej Ehrenfeucht and Grzegorz Rozenberg

CU-CS-190-80 December 1980

University of Colorado at Boulder
DEPARTMENT OF COMPUTER SCIENCE

GENERALIZED POST CORRESPONDENCE PROBLEM OF LENGTH 2
PART III. Decidability

by

A Ehrenfeucht*

and

G. Rozenberg**

CU-CS-190-80 September, 1980

*A. Ehrenfeucht, Department of Computer Science, University of Colorado at Boulder, Boulder, Colorado 80309 USA

**G. Rozenberg, Institute of Applied Mathematics and Computer Science, University of Leiden, 2300 RA Leiden, The Netherlands

All correspondence to G. Rozenberg

ABSTRACT

This paper forms the last part of the sequence of three papers which investigate the Generalized Post Correspondence Problem of length 2 (GPCP(2) for short). Using the results of the first two parts we demonstrate that GPCP(2) is decidable. As a corollary we get that Post Correspondence Problem of length two is decidable.

INTRODUCTION

In this paper we continue the investigation of the Generalized Post Correspondence Problem of length 2 (GPCP(2) for short) started in [ER1] and [ER2]. Using several reduction techniques we demonstrate that as far as the decidability status of GPCP(2) is concerned it suffices to consider a number of "special" cases investigated in [ER1] and [ER2]. Since the decidability of those special cases was demonstrated in [ER1] and [ER2] we are able to prove that GPCP(2) is decidable. The basic tool in our solution is the notion of a stable instance of GPCP(2) introduced in [ER1] and further investigated in [ER2].

We use the notation and terminology from the previous two parts. Perhaps the only new notation is the following one. For an instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2), Right(I) denotes the set $\{h(0),h(1),g(0),g(1)\}$.

Whenever we refer to a result from Part I or Part II of this paper we precede its "identification number" by I and II respectively; thus, e.g., Theorem I.3.1 refers to Theorem 3.1 from Part I.

1. FIRST REDUCTION THEOREM

In this section we demonstrate that in considering the decidability status of GPCP(2) one can restrict oneself to the investigation of two cases: periodic and marked instances of GPCP(2).

Theorem 1.1. There exists an algorithm which given an arbitrary instance I of GPCP(2) that is not periodic, produces a positive integer D and a finite set MAR(I) of marked instances of GPCP(2) such that I has a solution if and only if either I has a solution not longer than D, or there exists a $J \in$ MAR(I) such that J has a solution.

Proof.

Let $I = (h,g,a_1,a_2,b_1,b_2)$ be a non-periodic instance of GPCP(2). Since I is non-periodic, neither h nor g is a periodic homomorphism. Consequently $h(01) \neq h(10)$ and $g(01) \neq g(10)$. Let z be the maximal common prefix of h(01) and h(10) and let $|z| = m_h$; similarly let v be the maximal common prefix of g(01) and g(10) and let $|v| = m_g$. Let cyc_1 be the operation on $\{0,1\}^*$ such that for a nonempty word $w = cu$, $c \in \{0,1\}$, $u \in \{0,1\}^*$, $cyc_1(w) = uc$; also we set $cyc_1(\Lambda) = \Lambda$. Then for a positive integer k, cyc_k denotes the k-folded composition of cyc_1 with itself. For a homomorphism f of $\{0,1\}^*$ and a positive integer k we set $f_{[k]}$ to be equal to the composition of f and cyc_k; hence $f_{[k]}(w) = cyc_k(f(w))$ for every $w \in \{0,1\}$.

Let $a_1' = a_1 z$ and $b_1' = b_1 v$. Let r be the minimal positive integer such that for every word $x \in \{0,1\}^+$ with $|x| = r$, $|h(x)| \geq |h(01)|$ and $|g(x)| \geq |g(01)|$. For every word $u \in \{0,1\}^+$ such that $|u| = r$ let $a_{2,u} = (z \setminus h(u))a_2$ and $b_{2,u} = (v \setminus g(u))b_2$.

Let $W = \{(a_{2,u}, b_{2,u}) : u \in \{0,1\}^+$ and $|u| = r\}$. Let h' be defined by $h'(c) = \mathit{cyc}_{m_h}(h(c))$ and let g' be defined by $g'(c) = \mathit{cyc}_{m_g}(g(c))$ where $c \in \{0,1\}$. It is easily seen that

$$h' = h_{[m_h]} \text{ and } g' = g_{[m_g]} \quad \text{.......................................} (1.1)$$

Let $MAR(I) = \{(h', g', a_1', a_{2,u}, b_1', b_{2,u}) : (a_{2,u}, b_{2,u}) \in W\}$ and let $D = 2r$. It is easy to see that the theorem holds. The crucial observation is that, by (1.1), for $w,u \in \{0,1\}^+$ with $|w| \geq r$ and $|u| = r$ we have $a_1 h(wu) a_2 = a_1' h'(w) a_{2,u}$ and $b_1 g(wu) b_2 = b_1' g'(w) b_{2,u}$. □

2. SECOND REDUCTION THEOREM

In the last section we have demonstrated that in considering the decidability status of GPCP(2) it suffices to consider only periodic and marked instances of GPCP(2). Since it is decidable whether or not an arbitrary periodic instance of GPCP(2) has a solution, see Theorem I.2.1, in the next two sections we will consider marked instances of GPCP(2) only.

In this section we demonstrate that if one considers the decidability status of stable instances of GPCP(2) then it suffices to consider nine "quite concrete" cases. (In what follows, for a word x such that $|x| \geq 2$, we use *two*(x) to denote the prefix of x consisting of the first two letters of x.)

Theorem 2.1. There exists an algorithm which given an arbitrary stable instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2) decides whether or not it has a solution, unless I belongs to one of the following nine categories.

$I \in CAT_1$ if

$h(0) = 0$, $h(1) = 1\alpha$, where $\alpha \in \{0,1\}^+$, and

$g(0) = 0\beta$, $g(1) = 1$, where $\beta \in \{0,1\}^+$.

For $i \in \{0,1\}$, $I \in CAT_{2,i}$ if

$h(0) = 0$, $h(1) = 1\alpha$, where $\alpha \in \{0,1\}^+$, and

$g(i) = 0\beta$, $g(1-i) = 1\gamma$, where $\beta,\gamma \in \{0,1\}^+$.

For $i \in \{0,1\}$, $I \in CAT_{3,i}$ if

$two(h(0)) = 00$, $two(h(1)) = 10$,

$two(g(i)) = 00$, $two(g(1-i)) = 10$.

For $i \in \{0,1\}$, $I \in CAT_{4,i}$ if

$two(h(0)) = 01$, $two(h(1)) = 10$,

$two(g(i)) = 01$, $two(g(1-i)) = 10$.

For $i \in \{0,1\}$, $I \in CAT_{5,i}$ if
$two(h(0)) = 00$, $two(h(1)) = 11$,
$two(g(i)) = 00$, $two(g(1-i)) = 11$.

Proof.

Consider an arbitrary stable instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2).

We consider separately several cases.

(I). At least one element of Right(I) is of length 1.

(I.1). If at least three elements of Right(I) are of length 1 then by Theorem I.1.1 we can decide whether or not I has a solution.

(I.2). Assume that exactly two elements of Right(I) are of length 1.

(I.2.1). If either $|h(0)| = 1$ and $|h(1)| = 1$ or $|g(0)| = 1$ and $|g(1)| = 1$, then by Theorem I.1.1 we can decide whether or not I has a solution. Thus we have the following case.

(I.2.2). There exist $k,\ell \in \{0,1\}$ such that $|h(k)| = 1$ and $|g(\ell)| = 1$; if $h(k) = g(\ell)$ then by Theorem I.4.2 we can decide whether or not I has a solution. Hence we assume that $h(k) \neq g(\ell)$.

Let us consider h first.

Clearly we can assume that $h(0) = 0$ and $h(1) = 1\alpha$ for $\alpha \neq \Lambda$ (because other cases can be reduced to this one using appropriate switches).

Now if $g(0) = 1$ and $g(1) = 0\beta$ for $\beta \neq \Lambda$ then by Theorem I.1.1 we can decide whether or not I has a solution.

Thus we can assume that $g(0) = 0\beta$ and $g(1) = 1$ for $\beta \neq \Lambda$, and consequently $I \in CAT_1$.

(I.3). Assume that exactly one element of Right(I) is of length 1.

Clearly we can assume that h(0) = 0. Consequently we have h(1) = 1α for $\alpha \neq \Lambda$ and we have two possibilities for g:
either g(0) = 0β and g(1) = 1γ for $\beta \neq \Lambda$, $\gamma \neq \Lambda$,
or g(0) = 1γ and g(1) = 0β for $\beta \neq \Lambda$, $\gamma \neq \Lambda$.
If the first of these two cases holds then $I \in CAT_{2,0}$, otherwise $I \in CAT_{2,1}$.
(II). All elements of Right(I) are of length bigger than 1.

Clearly we can assume that *first*(h(0)) = 0 (otherwise we apply the range switch).

We have the following possibilities for the pair (*two*(h(0)),*two*(h(1))).
(a). *two*(h(0)) = 00 and *two*(h(1)) = 10,
(b). *two*(h(0)) = 01 and *two*(h(1)) = 10, and
(c). *two*(h(0)) = 00 and *two*(h(1)) = 11.
We do not have to consider the remaining possibility, *two*(h(0)) = 01 and *two*(h(1)) = 11, because applying the domain switch and the range switch we get then the case (a) above.
(II.1). Assume now that *first*(g(0)) = 0.
Then the reader can easily check that the application of Lemma I.3.6 yields the following:
if (a) holds then *two*(g(0)) = 00 and *two*(g(1)) = 10,
if (b) holds then *two*(g(0)) = 01 and *two*(g(1)) = 10, and
if (c) holds then *two*(g(0)) = 00 and *two*(g(1)) = 11.

Thus in this case $I \in CAT_{3,0}$ or $I \in CAT_{4,0}$ or $I \in CAT_{5,0}$ respectively.
(II.2). Assume that *first*(g(0)) = 1.

Then the reader can easily check that the application of Lemma I.3.6 yields the following:
if (a) holds then *two*(g(0)) = 10 and *two*(g(1)) = 00,
if (b) holds then *two*(g(0)) = 10 and *two*(g(1)) = 01, and

if (c) holds then *two*(g(0)) = 11 and *two*(g(1)) = 00.

Thus in this case I $\in$ $CAT_{3,1}$ or I $\in$ $CAT_{4,1}$ or I $\in$ $CAT_{5,1}$ respectively.

Consequently Theorem 2.1 holds. □

3. THIRD REDUCTION THEOREM

In this section we continue the investigation of stable instances of GPCP(2). In particular, starting with the nine cases described in the last section, we demonstrate that in considering the decidability status of GPCP(2) it suffices to consider six categories which quite precisely describe the exact pattern of images of homomorphisms involved in an instance of GPCP(2). Since these six categories were investigated in Part II of this paper we will be ready to settle the decidability of GPCP(2) in the next section.

We start by recalling from Part II the definition of six (regular) languages.

For $i \in \{0,1\}$, $A_i = i^+$, $B_i = i(1-i)^*$ and $C_i = i((1-i)i)^*\{\Lambda,(1-i)\}$.

Theorem 3.1. There exists an algorithm which given an arbitrary stable instance $I = (h,g,a_1,a_2,b_1,b_2)$ of GPCP(2) decides whether or not I has a solution, unless I belongs to one of the following six categories.

For $i \in \{0,1\}$, $I \in CL_{A_i}$ if

$h(0) \in A_0$, $h(1) \in A_1$, $g(i) \in A_0$ and $g(1-i) \in A_1$.

For $i \in \{0,1\}$, $I \in CL_{B_i}$ if

$h(0) \in A_0$, $h(1) \in B_1$, $g(i) \in A_0$ and $g(1-i) \in B_1$.

For $i \in \{0,1\}$, $I \in CL_{C_i}$ if

$h(0) \in C_0$, $h(1) \in C_1$, $g(i) \in C_0$ and $g(1-i) \in C_1$.

Proof.

By Theorem 2.1 we may assume that I belongs to one of the nine categories: CAT_1, $CAT_{2,i}$, $CAT_{3,i}$, $CAT_{4,i}$, or $CAT_{5,i}$ for $i \in \{0,1\}$. We will consider those categories separately.

(a). Assume that $I \in CAT_{5,i}$ for $i \in \{0,1\}$.

Hence *two*(h(0)) = 00, *two*(h(1)) = 11, *two*(g(i)) = 00 and *two*(g(1-i)) = 11.

Assume that $h(0) \notin 0^+$. Then $01\alpha \in$ *suf*(h(0)) for some $\alpha \in \{0,1\}^*$. However 01α is not prefix compatible with any element of {g(0),g(1)} which contradicts Lemma I.3.6.

Consequently $h(0) \in 0^+$; hence $h(0) \in A_0$.

Assume that $h(1) \notin 1^+$. Then $10\alpha \in$ *suf*(h(1)) for some $\alpha \in \{0,1\}^*$. However 10α is not prefix compatible with any element of {g(0),g(1)} which contradicts Lemma I.3.6.

Consequently $h(1) \in 1^+$; hence $h(1) \in A_1$.

Similarly we show that $g(i) \in A_0$ and $g(1-i) \in A_1$.

Thus $I \in CL_{A_i}$.

(b). Assume that $I \in CAT_{4,i}$ for $i \in \{0,1\}$.

Hence *two*(h(0)) = 01, *two*(h(1)) = 10, *two*(g(i)) = 01 and *two*(g(1-i)) = 10.

Assume that $h(0) \notin C_0$. Then either $00\alpha \in$ *suf*(h(0)) or $11\alpha \in$ *suf*(h(0)) for some $\alpha \in \{0,1\}^*$. However neither $\{00\alpha\}$ nor $\{11\alpha\}$ is prefix compatible with {g(0),g(1)} which contradicts Lemma 1.3.6.

Consequently $h(0) \in C_0$.

Similarly we prove that $h(1) \in C_1$, $g(i) \in C_0$ and $g(1-i) \in C_1$.

Thus $I \in CL_{C_i}$.

(c). Assume that $I \in CAT_{3,i}$ for $i \in \{0,1\}$.

Hence *two*(h(0)) = 00, *two*(h(1)) = 10, *two*(g(i)) = 00 and *two*(g(1-i)) = 10.

Similarly to cases (a) and (b) we prove then that $h(0) \in A_0$, $h(1) \in B_1$, $g(i) \in A_0$ and $g(1-i) \in B_1$. Consequently $I \in CL_{B_i}$.

(d). Assume that $I \in CAT_{2,i}$ for $i \in \{0,1\}$.

Hence $h(0) = 0$, $h(1) = 1\alpha$, $g(i) = 0\beta$ and $g(1-i) = 1\gamma$ where $\alpha,\beta,\gamma \in \{0,1\}^+$.

If $|\alpha| = 1$ then $|h(0)| \leq |g(0)|$ and $|h(1)| \leq |g(1)|$ and so by Theorem I.1.1 it is decidable whether or not I has a solution.

Thus we may assume that $|\alpha| \geq 2$.

Then if we consider all possibilities for $(\mathit{first}(\alpha),\mathit{first}(\beta))$ we get four cases, which we will consider now.

(d.1). $\mathit{first}(\alpha) = 0$ and $\mathit{first}(\beta) = 0$.

Hence $h(0) = 0$, $h(1) = 10\alpha'$, $g(i) = 00\beta'$ and $g(1-i) = 1\gamma$ where $\alpha',\beta',\gamma \in \{0,1\}^*$, $\alpha' \neq \Lambda$ and $\gamma \neq \Lambda$.

Observe that $\alpha' \in 0^*$ as otherwise $01\rho \in \mathit{suf}(h(0))$ for some $\rho \in \{0,1\}^*$; but $\{01\rho\}$ is not prefix compatible with $\{g(0),g(1)\}$ which contradicts Lemma I.3.6.

Assume now that $\beta' \notin 0^*$.

Let x be a word over $\{0,1\}$. Then

$$\#_1 g(x) = \#_1 g(0)\#_0 x + \#_1 g(1)\#_1 x \qquad (3.1)$$

Since $\beta' \notin 0^*$,

$$\#_1 g(i) \neq 0 \text{ and } \#_1 g(1-i) \neq 0 \qquad (3.2)$$

Since $\alpha' \in 0^*$

$$\#_1 h(x) = \#_1 x \qquad (3.3)$$

Let $t_1(x) = \#_1 g(x) - \#_1 h(x)$.

Then (3.1) and (3.3) implies that

$$t_1(x) = \#_1 g(0)\#_0 x + (\#_1 g(1) - 1)\#_1 x \qquad (3.4)$$

Assume now that x is a solution of I. Then, obviously,

$$t_1(x) \leq \#_1 a_1 a_2 b_1 b_2 \qquad (3.5)$$

and so (3.4) implies that

$$\#_0 x \leq \frac{\#_1 a_1 a_2 b_1 b_2 - (\#_1 g(1) - 1)\#_1 x}{\#_1 g(0)} \leq \frac{\#_1 a_1 a_2 b_1 b_2}{\#_1 g(0)}$$

(note that, by (3.2), $\#_1 g(0) \neq 0$).

Thus, by Theorem I.1.2, it is decidable whether or not I has a solution.

So we can assume now that $\beta' \in 0^*$.

Thus $h(0) = 0$, $h(1) = 10^n$, $n \geq 2$ and $g(i) = 0^m$, $m \geq 2$.

From Lemma I.3.6 it easily follows that

$g(1-i) = 10^{k_1}10^{k_2} \ldots 10^{k_\ell}$ for some $\ell \geq 1$,

where if $\ell > 1$ then $k_j \geq n$ for $j \in \{1,\ldots,(\ell-1)\}$.

We consider separately two cases.

<u>$\ell = 1$</u>.

Then $I \in CL_{B_i}$.

<u>$\ell \geq 2$</u>.

Here we consider separately two cases.

<u>$i = 0$</u>.

Then we have $h(0) = 0$, $h(1) = 10^n$, $n \geq 2$, $g(0) = 0^m$, $m \geq 2$ and $g(1) = 10^{k_1}10^{k_2}1 \ldots 10^{k_\ell}$, $\ell \geq 2$ where $k_j \geq n$ for $j \in \{1,\ldots,(\ell-1)\}$.

Thus by Theorem I.1.1 it is decidable whether or not I is solvable.

i = 1.

Then we have $h(0) = 0$, $h(1) = 10^n$, $n \geq 2$, $g(1) = 0^m$, $m \geq 2$ and $g(0) = 10^{k_1}10^{k_2}1 \ldots 10^{k_\ell}$, $\ell \geq 2$ where $k_j \geq n$ for $j \in \{1,\ldots,(\ell-1)\}$.

Construct $ecol(h,g) = (\overline{h},\overline{g})$.

It is easy to see that

$\overline{h}(0) = 0^m$, $\overline{g}(0) = 1$, $\overline{h}(1) = 1\delta$ for some $\delta \in \{0,1\}^+$ and $\overline{g}(1) = 01^r$ for some $r \geq 0$.

If $r = 0$, then $|\overline{g}(0)| \leq |\overline{h}(0)|$ and $|\overline{g}(1)| \leq |\overline{h}(1)|$, and so by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 it is decidable whether or not I has a solution.

However if $r \geq 1$, then $01^r \in suf(\overline{g}(1))$ while $\{01^r\}$ is not prefix compatible with $\{\overline{h}(0),\overline{h}(1)\}$; this contradicts Lemma 1.3.6 and so it must be that $r = 0$.

(d.2). $first(\alpha) = 0$ and $first(\beta) = 1$.

Hence $h(0) = 0$, $h(1) = 10\alpha'$, $g(i) = 01\beta'$ and $g(1-i) = 1\gamma$ where $\alpha',\beta',\gamma \in \{0,1\}^*$, $\alpha' \neq \Lambda$ and $\gamma \neq \Lambda$.

It is easily seen that by Lemma I.3.6, $\gamma = 0\gamma'$ for some $\gamma' \in \{0,1\}^*$.

Lemma 1.3.6 implies that $\alpha' = 1\alpha''$ for some $\alpha'' \in \{0,1\}^*$.

Thus we have $h(0) = 0$, $h(1) = 101\alpha''$, $g(i) = 01\beta'$ and $g(1-i) = 10\gamma'$.

Then it is easily seen that, by Lemma I.3.6, $h(0) \in C_0$, $h(1) \in C_1$, $g(i) \in C_0$ and $g(1-i) \in C_1$.

Consequently $I \in CL_{C_i}$.

(d.3). *first*(α) = 1 and *first*(β) = 0.

Hence $h(0) = 0$, $h(1) = 11\alpha'$, $g(i) = 00\beta'$ and $g(1-i) = 1\gamma$ for some $\alpha', \beta', \gamma \in \{0,1\}^*$, $\alpha' \neq \Lambda$ and $\gamma \neq \Lambda$.

It is easily seen that by Lemma I.3.6, $\gamma \in 1^+$. Then, again by Lemma I.3.6, $\alpha' \in 1^+$.

By Lemma I.3.6, $\beta' \in 0^*1^*$ and consequently we have $h(0) = 0$, $h(1) = 1^\ell$, $\ell \geq 3$, $g(i) = 0^r1^s$, $r \geq 2$, $s \geq 0$ and $g(1-i) = 1^n$, $n \geq 2$.

We consider separately two cases.

<u>$i = 0$.</u>

Then we have $h(0) = 0$, $h(1) = 1^\ell$, $\ell \geq 3$, $g(0) = 0^r1^s$, $r \geq 2$, $s \geq 0$ and $g(1) = 1^n$, $n \geq 2$.

Let x be a word over $\{0,1\}$. Let

$t_0(x) = \#_0 g(x) - \#_0 h(x)$.

Then we have

$$t_0(x) = (r-1)\#_0 x \quad \text{...} \quad (3.6)$$

Assume now that x is a solution of I. Then, obviously,

$t_0(x) \leq \#_0 a_1a_2b_1b_2$

and so, by (3.6),

$$\#_0 x \leq \frac{\#_0 a_1a_2b_1b_2}{r-1}$$

(note that since $r \geq 2$, $r-1 \neq 0$).

Thus, by Theorem I.1.2 we can decide whether or not I has a solution.

<u>$i = 1$.</u>

Then we have $h(0) = 0$, $h(1) = 1^\ell$, $\ell \geq 3$, $g(0) = 1^n$, $n \geq 2$ and $g(1) = 0^r1^s$, $r \geq 2$, $s \geq 0$.

If $s = 0$ then $h(0) \in A_0$, $h(1) \in A_1$, $g(0) \in A_1$ and $g(1) \in A_0$. Consequently $I \in CL_{A_1}$.

So let us assume that $s \geq 1$.

Construct $ecol(h,g) = (\overline{h},\overline{g})$. It is easily seen that $\overline{h}(0) = 0^r1^z$ for some $z \geq 1$, $\overline{h}(1) \in 1^+$, $\overline{g}(0) \in 10^*$ and $\overline{g}(1) \in 0^+$.

But if $|\overline{g}(1)| \geq 2$ then $01^z \in suf(\overline{h}(0))$ while $\{01^z\}$ is not prefix compatible with $\{\overline{g}(0),\overline{g}(1)\}$, which contradicts Lemma 1.3.6.

Consequently $\overline{g}(1) = 0$. If $|\overline{g}(0)| = 1$ then $|\overline{g}(0)| \leq |\overline{h}(0)|$ and $|\overline{g}(1)| < |\overline{h}(1)|$; thus by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 it is decidable whether or not I has a solution. If $|\overline{g}(0)| \geq 2$ then by Lemma I.3.6 it must be that $\overline{h}(1) = 1$; thus, again, by Theorem I.1.1, Theorem I.3.1 and Theorem I.4.1 it is decidable whether or not I has a solution.

(d.4). $first(\alpha) = 1$ and $first(\beta) = 1$.

Then we have $h(0) = 0$, $h(1) = 11\alpha'$, $g(i) = 01\beta'$ and $g(1-i) = 1\gamma$ for some $\alpha',\beta',\gamma \in \{0,1\}^*$, $\alpha' \neq \Lambda$ and $\gamma \neq \Lambda$.

It is easily seen that, by Lemma I.3.6, it must be that $\gamma \in 1^+$, $\beta' \in 1^*$ and $\alpha' \in 1^+$. Consequently, $h(0) = 0$, $h(1) = 1^n$, $n \geq 3$, $g(i) = 01^m$, $m \geq 1$ and $g(1-i) = 1^r$, $r \geq 2$. Applying the domain switch and the range switch we get then that $I \in CL_{B_i}$.

(e). Assume that $I \in CAT_1$.

Hence $h(0) = 0$, $h(1) = 1\alpha$, $g(0) = 0\beta$ and $g(1) = 1$ where $\alpha,\beta \in \{0,1\}^+$.

Then, if we consider all possibilities for $(first(\alpha),first(\beta))$, we get four cases which we will consider now.

(e.1). $first(\alpha) = 0$ and $first(\beta) = 0$.

Hence $h(0) = 0$, $h(1) = 10\alpha'$, $g(0) = 00\beta'$ and $g(1) = 1$ where $\alpha',\beta' \in \{0,1\}^*$.

By Lemma I.3.6, $\alpha' \in 0^*$.

As far as β' is concerned we have two possibilities.

$\underline{\#_1\beta' > 0}$.

Let x be a word over $\{0,1\}$. Then

$\#_1h(x) = \#_1x$ and $\#_1g(x) = \#_1x + \#_1g(0)\#_0x$.

Consequently

$t_1(x) = \#_1 g(x) - \#_1 h(x) = \#_1 g(0)\#_0 x = \#_1 \beta'\#_0 x.$

Assume now that x is a solution of I. Then, obviously,

$t_1(x) \leq \#_1 a_1 a_2 b_1 b_2$

and so

$$\#_0 x \leq \frac{\#_1 a_1 a_2 b_1 b_2}{\#_1 \beta'}.$$

Consequently, by Theorem I.1.2, we can decide whether or not I has a solution.

$\underline{\#_1 \beta' = 0.}$

Then $h(0) = 0$, $h(1) = 10^n$, $n \geq 1$, $g(0) = 0^m$, $m \geq 2$ and $g(1) = 1$; hence $I \in CL_{B_0}$.

(e.2). $\mathit{first}(\alpha) = 1$ and $\mathit{first}(\beta) = 1$.

Hence $h(0) = 0$, $h(1) = 11\alpha'$, $g(0) = 01\beta'$ and $g(1) = 1$ where $\alpha', \beta' \in \{0,1\}^*$.

It is easily seen that applying the domain switch and the range switch and the homomorphisms switch one obtains case (e.1).

(e.3). $\mathit{first}(\alpha) = 0$ and $\mathit{first}(\beta) = 1$.

Hence we have $h(0) = 0$, $h(1) = 10\alpha'$, $g(0) = 01\beta'$ and $g(1) = 1$ where $\alpha', \beta' \in \{0,1\}^*$.

We consider separately three cases.

$\underline{\alpha' \neq \Lambda \text{ and } \beta' \neq \Lambda.}$

Applying (iteratively) Lemma I.3.6 one can easily see that then $h(1) \in C_1$ and $g(0) \in C_0$; consequently $I \in CL_{C_0}$.

$\underline{\alpha' = \Lambda.}$

Hence $h(0) = 0$, $h(1) = 10$, $g(0) = 01\beta'$ and $g(1) = 1$.

Let x be a word over {0,1}. Then we have

$\#_1 h(x) = \#_1 x$ and $\#_1 g(x) = \#_1 x + \#_1 g(0)\#_0 x$.

Thus

$t_1(x) = \#_1 g(x) - \#_1 h(x) = \#_1 g(0)\#_0 x.$

Assume now that x is a solution of I. Then, obviously,

$t_1(x) \leq \#_1 a_1 a_2 b_1 b_2$

and so

$$\#_0 x \le \frac{\#_1 a_1 a_2 b_1 b_2}{\#_1 g(0)}.$$

Consequently, by Theorem I.1.2, we can decide whether or not I has a solution.

<u>$\beta' = \Lambda$.</u>

Hence $h(0) = 0$, $h(1) = 10\alpha'$, $g(0) = 01$ and $g(1) = 1$.

It is easily seen that applying then the domain switch, the range switch and the homomorphisms switch one gets case $\alpha' = \Lambda$.

(e.4). *first*$(\alpha) = 1$ and *first*$(\beta) = 0$.

Hence $h(0) = 0$, $h(1) = 11\alpha'$, $g(0) = 00\beta'$ and $g(1) = 1$ for some $\alpha', \beta' \in \{0,1\}^*$.

It is easily seen that, by Lemma I.3.6, $\alpha' \in 1^*0^*$ and $\beta' \in 0^*1^*$. Thus we have $h(0) = 0$, $h(1) = 1^k 0^\ell$, $k \ge 2$, $g(0) = 0^m 1^n$, $m \ge 2$ and $g(1) = 1$.

We note that if both $\ell \ne 0$ and $n \ne 0$ then we have $0 = \mathit{last}(h(0)) = \mathit{last}(h(1)) \ne \mathit{last}(g(0)) = \mathit{last}(g(1)) = 1$ which contradicts Lemma I.3.7.

Thus we have three cases to consider.

<u>$\ell = 0$ and $n \ne 0$.</u>

Then $h(0) = 0$, $h(1) = 1^k$, $g(0) = 0^m 1^n$ and $g(1) = 1$.

Let x be a word over $\{0,1\}$. Then we have

$\#_0 h(x) = \#_0 x$ and $\#_0 g(x) = m\#_0 x$.

If x is a solution of I then we have

$t_0(x) = \#_0 g(x) - \#_0 h(x) \le \#_1 a_1 a_2 b_1 b_2$

and consequently

$$\#_0 x \quad \frac{\#_0 a_1 a_2 b_1 b_2}{m-1}$$

(note that $m \ge 2$ and so $(m-1) \ne 0$).

Thus, by Theorem I.1.2, it is decidable whether or not I has a solution.

<u>$n = 0$ and $\ell \neq 0$.</u>

Then $h(0) = 0$, $h(1) = 1^k 0^\ell$, $g(0) = 0^m$ and $g(1) = 1$.

It is easily seen that applying the domain switch, the range switch and the homomorphisms switch one gets the case $n \neq 0$ and $\ell = 0$.

<u>$n = 0$ and $\ell = 0$.</u>

Then $h(0) = 0$, $h(1) = 1^k$, $g(0) = 0^m$ and $g(1) = 1$; hence $I \in CL_{A_0}$.

This completes our proof of the theorem. □

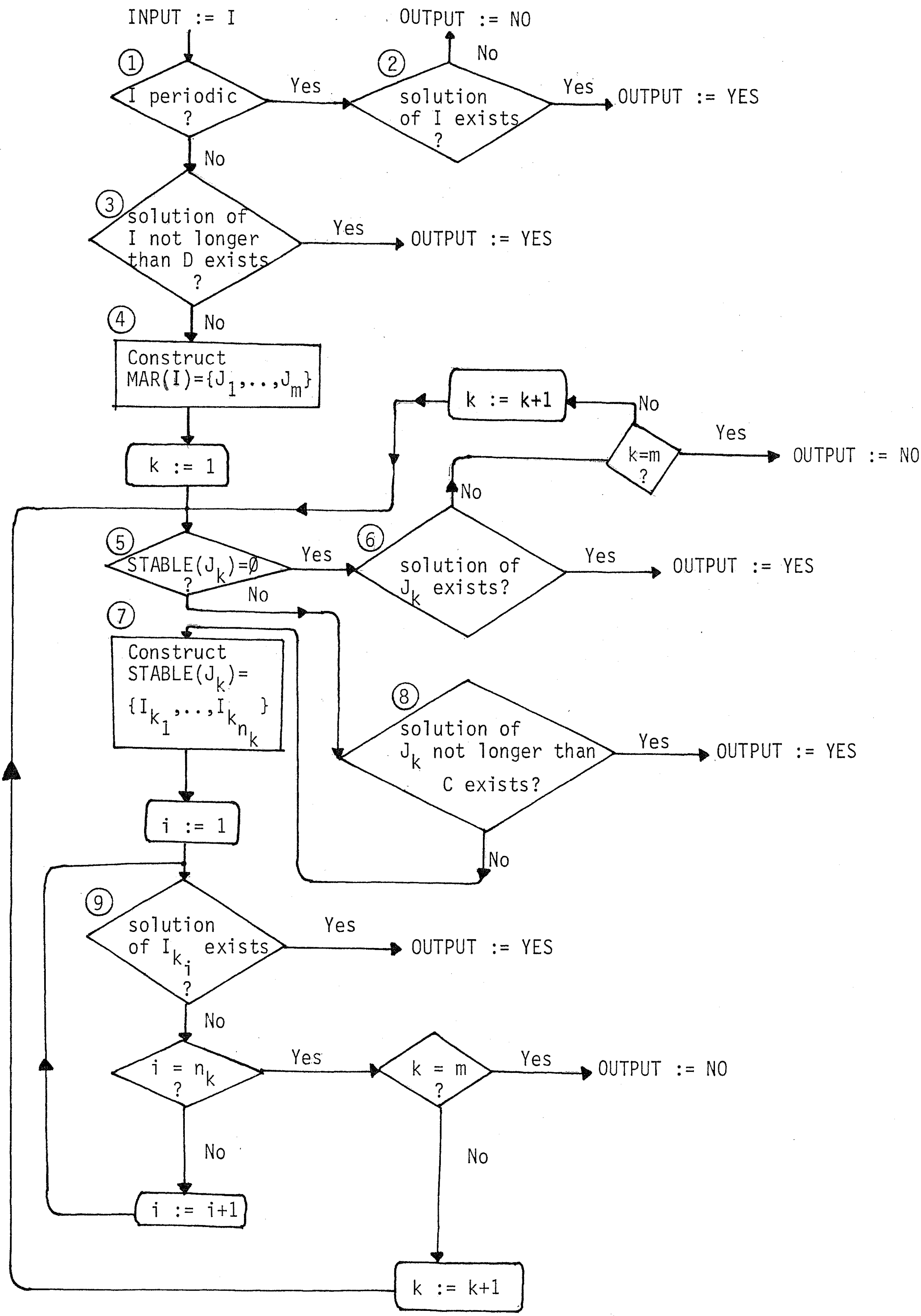
4. THE MAIN THEOREM

In this section we demonstrate that GPCP(2) is decidable. The algorithm deciding whether or not an arbitrary instance I of GPCP(2) has a solution uses algorithms from Part I and Part II of this paper which settle the decidability of various subcases; the validity of the algorithm follows essentially from the reduction theorems presented in this part of the paper.

Theorem 4.1. It is decidable whether or not an arbitrary instance of GPCP(2) has a solution

Proof.

The following algorithm given an arbitrary instance I of GPCP(2) gives answer YES is I has a solution and answer NO if I has no solution. (In the following flowchart of our algorithm, D is the constant from the statement of Theorem 1.1 and C is the constant from the statement of Theorem I.3.1; the set MAR(I) is the set referred to in the statement of Theorem 1.1).



INPUT := I
1
I periodic ?
Yes
No
2
solution of I exists ?
No
OUTPUT := NO
Yes
OUTPUT := YES
3
solution of I not longer than D exists ?
Yes
OUTPUT := YES
No
4
Construct MAR(I)={J_1,...,J_m}
k := 1
k := k+1
No
k=m ?
Yes
OUTPUT := NO
No
5
STABLE(J_k)=∅ ?
Yes
6
solution of J_k exists?
Yes
OUTPUT := YES
No
7
Construct STABLE(J_k)= {I_k_1,...,I_k_n_k}
8
solution of J_k not longer than C exists?
Yes
OUTPUT := YES
No
i := 1
9
solution of I_k_i exists ?
Yes
OUTPUT := YES
No
i = n_k ?
Yes
k = m ?
Yes
OUTPUT := NO
No
i := i+1
No
k := k+1

Test 1 is obviously effective.

Test 2 is effective by Theorem I.2.1.

Test 3 is obviously effective.

Construction 4 is obviously effective.

Test 5 is effective by Lemma I.3.8.

Test 6 is effective by Theorem I.4.1.

Construction 7 is effective by Lemma I.3.8.

Test 8 is obviously effective.

Test 9 is effective by Theorem 3.1 and Theorem II.7.1.

The correctness of the algorithm follows from Theorem I.3.1, Theorem I.4.1, Theorem II.7.1, Theorem 1.1 and Theorem 3.1. □

Corollary 4.1. It is decidable whether or not an arbitrary instance of PCP(2) has a solution.

Proof.

PCP(2) is a special case of the GPCP(2). □

5. DISCUSSION

REFERENCES

[ER1]. Ehrenfeucht, A. and Rozenberg, G., Generalized Post Correspondence Problem of length 2. Part I. Some special cases and the basic transformation. University of Colorado at Boulder, Department of Computer Science, Technical Report No. CU-CS-188-80, 1980.

[ER2]. Ehrenfeucht, A. and Rozenberg, G., Generalized Post Correspondence Problem of length 2. Part II. Cases distinguished by patterns. University of Colorado at Boulder, Department of Computer Science, Technical Report No. CU-CS-189-80, 1980.

ACKNOWLEDGMENTS

The authors gratefully acknowledge support under National Science Foundation grant number MCS 79-03838.