ON UNARY 2-FOLD EQUATIONS

by

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ABSTRACT

Let Σ, Δ be (finite) alphabets where Σ consists of one letter only. Let f_1, f_2, g_1, g_2 be homomorphisms from Σ^* into Δ^* and let $a_1, a_2, a_3, b_1, b_2, b_3$ be words over Δ. We demonstrate that it is decidable whether or not the equation $a_1 f_1(x_1) a_2 f_2(x_2) a_3 = b_1 g_1(x_1) b_2 g_2(x_2) b_3$ has a solution.

INTRODUCTION

Equations in free monoids are used quite frequently in solving various problems of formal language theory, see, e.g., [La], [Le] and [S]. In particular equations involving homomorphisms of free monoids are met frequently in the theory of L systems and the theory of equality languages which investigates solutions of Post Correspondence Problem and its generalizations (see, e.g., [RS]).

In this paper we investigate the following type of equations. Let $f_1,\ldots,f_n,g_1,\ldots,g_n$ be homomorphisms from Σ^* into Δ^* where Σ consists of one letter only and let $a_1,\ldots,a_{n+1},b_1,\ldots,b_{n+1}$ be words over Δ. Then the equation of the form $a_1f_1(x_1)a_2f_2(x_2)\ldots a_nf_n(x_n)a_{n+1} = b_1g_1(x_1)b_2g_2(x_2)\ldots b_ng_n(x_n)b_{n+1}$ is called a unary n-fold equation. We demonstrate that the problem "Does an arbitrary unary 2-fold equation have a solution?" is decidable, and we argue that the same problem for an arbitrary unary n-fold equation is decidable. Our result turns out to be quite important in demonstrating that the Post Correspondence Problem for lists of length two is decidable (see [ER]).

I. PRELIMINARIES

We use standard language-theoretic notation and terminology. Perhaps only the following points require additional comments.

In this paper we consider finite alphabets only.

For an integer n, *abs*(n) denotes the absolute value of n.

For a finite set Z, #Z denotes its cardinality.

For a word x, $|x|$ denotes its length; Λ denotes the empty word.

For words x and y we write x *sub* y, x *pref* y and x *suf* y if x is a subword of y, a prefix of y or a suffix of y respectively (x is a subword of y if there exist words y_1,y_2 such that $y = y_1 x y_2$; y is its own prefix and its own suffix). For a nonnegative integer m, we use $pref_m(y)$ to denote the prefix of y of length m. If x *pref* y then $x \backslash y$ denotes the word z such that xz = y. We say that x and y are *cyclic conjugates* of each other if for some words w and z we have x = wz and y = zw; we write then $x \sim y$.

In this paper we consider propagating homomorphisms only (a homomorphism is called propagating if for no letter a, $h(a) = \Lambda$).

We recall now a basic combinatorial result, see, e.g., [H].

Lemma 0. Let $u,v \in \Delta^*$. Then $u = w^m$ and $v = w^n$ for some $w \in \Delta^*$, $m, n \geq 0$ if and only if there exist $p,q \geq 0$ so that u^p and v^q contain a common prefix (suffix) of length $|u| + |v| - z$ where z is the greatest common divisor of $|u|$ and $|v|$. □

The following notion is quite basic for this paper.

Definition. Let f, g be homomorphisms, $f, g : \Sigma^* \to \Delta^*$ where $\#\Sigma = 1$, say $\Sigma = \{0\}$. We say that f and g *agree*, and write them f *agr* g, if there exist $\alpha_1,\alpha_2 \in \Delta^*$ and positive integers m, n such that $f(0) = (\alpha_1\alpha_2)^m$ and $g(0) = (\alpha_2\alpha_1)^n$. Otherwise we say that f and g *disagree* and write f *dagr* g. □

It is not difficult to see that *agr* is an equivalence relation. The following property of the *agr* relation will be useful in the sequel.

Lemma 1. Let f, g be homomorphisms, $f, g : \Sigma^* \to \Delta^*$ where $\#\Sigma = 1$. If f *dagr* g and f(x) *sub* g(y) for $x, y \in \Sigma^*$, then $|x| \leq \max\{2, 2\frac{|g(0)|}{|f(0)|}\}$.

Proof.

Assume that f(x) *sub* g(y) where $|x| > \max\{2, 2\frac{|g(0)|}{|f(0)|}\}$ and so $|f(x)| > 2|g(0)|$. Hence there is a cyclic conjugate α of g(0) such that α^p and $f(0)^q$ where $p,q > 2$ contain a common prefix longer than $|\alpha| + |f(0)|$. Thus by Lemma 0, $\alpha = w^m$ and $f(0) = w^n$ for some $w \in \Delta^+$ and $m,n \geq 1$ which implies that f *agr* g; a contradiction. □

In this paper we investigate the following equations.

Definition. Let n be a positive integer, Σ and Δ alphabets where $\#\Sigma = 1$. Let $f_1,\ldots,f_n,g_1\ldots,g_n$ be homomorphisms from Σ^* into Δ^* and let $a_1,\ldots,a_{n+1},b_1,\ldots,b_{n+1} \in \Delta^*$. Then an *unary* n-*fold equation* is an equation of the form

$$a_1f_1(x_1)a_2f_2(x_2)\ldots.f_n(x_n)a_{n+1} = b_1g_1(x_1)b_2g_2(x_2)\ldots g_n(x_n)b_{n+1}\ldots..(E_n)$$

in variables $x_1,\ldots,x_n$.

A *solution* of (E_n) is a vector $(\alpha_1,\ldots.,\alpha_n)$ with $\alpha_1,\ldots,\alpha_n \in \Sigma^*$ such that $a_1f_1(\alpha_1)a_2f_2(\alpha_2)\ldots f_n(\alpha_n)a_{n+1} = b_1g_1(\alpha_1)b_2g_2(\alpha_2)\ldots g_n(\alpha_n)b_{n+1}$.

If (E_n) has a solution we say that it is *solvable*. □

The following decision problem will be the subject of investigation of this paper.

Definition. The *unary* n-*fold solvability problem* is defined as follows: "Let (E_n) be an arbitrary unary n-fold equation. Is (E_n) solvable?" □

II. UNARY 1-FOLD EQUATIONS

In this section we will demonstrate that the unary 1-fold solvability problem is decidable.

Theorem 1. It is decidable whether or not an arbitrary unary 1-fold equation has a solution.

Proof.

Consider a 1-fold equation

$$a_1f(x)a_2 = b_1g(x)b_2 \quad \ldots\ldots (E_1)$$

where f,g are homomorphisms from Σ^* into Δ^*, $\Sigma = \{0\}$ and $a_1,a_2,b_1,b_2 \in \Delta^*$. Clearly it is decidable whether or not $x = \Lambda$ is a solution of E_1. So we will assume in the rest of this section that $x \neq \Lambda$.

We investigate separately the following cases.

(1). f *agr* g.

Let us assume that $x = \alpha$ is a solution of (E_1), and let us consider the following two cases.

(1.1). $|f(\alpha)| \neq |g(\alpha)|$.

Let $d_1 = |a_1f(\alpha)a_2|$ and $d_2 = |b_1g(\alpha)b_2|$.

Since α is a solution of (E_1), $d_1 = d_2$ and consequently

$$|\alpha| = \frac{|b_1b_2| - |a_1a_2|}{|f(0)| - |g(0)|},$$

(notice that $|f(0)| - |g(0)| \neq 0$, because otherwise one gets $|f(\alpha)| = |g(\alpha)|$; a contradiction)

Thus in this case if a solution of (E_1) exists then it is unique and can be effectively found.

(1.2). $|f(\alpha)| = |g(\alpha)|$

Since f *agr* g, this implies that $f(0) \sim g(0)$.

Let $q_1 = |a_1|, \overline{q}_1 = |b_1|$, $q = \max\{q_1, \overline{q}_1\}$, $p_1 = |a_2|$, $\overline{p}_1 = |b_2|$, $p = \max\{p_1, \overline{p}_1\}$ and let $r = |f(0)| = |g(0)|$. Let $\delta = a_1 f(\alpha) a_2 = b_1 g(\alpha) b_2$ and let $t = q + p + 3r$.

Assume that $\delta = X_1 \dots X_s$ where $s > t$ and $X_1, \dots, X_s \in \Delta$; also let $\alpha = 0^m$ for some $m \geq 1$ (notice that $s > t$ implies that $m > 3$). Let $u_1 = s - p_1$, $\overline{u}_1 = s - \overline{p}_1$ and $u = \min\{u_1, \overline{u}_1\}$. Let n be the smallest positive integer such that $q_1 + nr > q$ and let $\overline{n}$ be the smallest positive integer such that $\overline{q}_1 + \overline{n}r > q$. Clearly $q_1 + (n+1)r < u$, $\overline{q}_1 + (\overline{n}+1)r < u$ and moreover (we assume that $q_1 + nr \leq \overline{q}_1 + \overline{n}r$; the other case can be considered in an analogous way)

$$X_1 \dots X_{q_1+nr} X_{q_1+(n+1)r+1} X_{q_1+(n+1)r+2} \dots X_s =$$

$$X_1 \dots X_{\overline{q}_1+\overline{n}r} X_{\overline{q}_1+(\overline{n}+1)r+1} X_{\overline{q}_1+(\overline{n}+1)r+2} \dots X_s .$$

Thus

$$a_1 f(0^{m-1}) a_2 = b_1 g(0^{m-1}) b_2 .$$

Consequently $x = 0^{m-1}$ is also a solution of (E_1).

Hence we have just shown that if (E_1) has a solution $\alpha = 0^m$ such that $|f(\alpha)| = |g(\alpha)|$ then (E_1) has such a solution where $|a_1 f(\alpha) a_2| \leq t$. Thus, also in this case, one can decide whether or not (E_1) has a solution and if (E_1) is solvable one can effectively find a solution of it.

(2). f *dagr* g

Let us assume that α is a solution of (E_1). If α is such that both $|f(\alpha)|$ and $|g(\alpha)|$ are longer than

$$t = abs(|a_1|-|b_1|) + abs(|a_2|-|b_2|) + (2\max\{|f(0)|,|g(0)|\})^2$$

then there exist words $x, y \in \Sigma^*$ such that either $f(x)$ *sub* $g(y)$ where $|x| > \max\{2, 2\frac{|g(0)|}{|f(0)|}\}$, or $g(x)$ *sub* $f(y)$ where $|x| > \max\{2, 2\frac{|f(0)|}{|g(0)|}\}$, which contradicts Lemma 1 (note that $\max\{2, 2\frac{|f(0)|}{|g(0)|}\} \leq 2\max\{|f(0)|, |g(0)|\}$.

Thus if α is a solution of (E_1) then either $|f(\alpha)| \leq t$ or $|g(\alpha)| \leq t$. Consequently also in this case it is decidable whether or not (E_1) is solvable; moreover if (E_1) is solvable one can effectively find a solution (as a matter of fact, all solutions) of it.

The theorem follows from (1) and (2). □

III. UNARY 2-FOLD EQUATIONS

In this section we will demonstrate that the unary 2-fold solvability problem is decidable.

Consider a unary 2-fold equation

$$a_1f_1(x)a_2f_2(y)a_3 = b_1g_1(x)b_2g_2(y)b_3 \quad \ldots\ldots (E_2)$$

where f_1, f_2, g_1, g_2 are homomorphisms from Σ^* into Δ^*, $\Sigma = \{0\}$ and $a_1, a_2, a_3, b_1, b_2, b_3 \in \Delta^*$.

Clearly if we look for a solution of (E_2) of the form $(x,y) = (\Lambda,\beta)$ or $(x,y) = (\alpha,\Lambda)$ then the problem reduces to a unary 1-fold equation. Hence in the rest of this section we assume that we are interested in solutions (α,β) of (E_2) where $\alpha \neq \Lambda$ and $\beta \neq \Lambda$.

We start by considering a special case of (E_2).

Lemma 2. Assume that there effectively exists a positive integer constant C such that for every solution (α,β) of (E_2) we have $abs\ (|a_1f_1(\alpha)a_2| - |b_1g_1(\alpha)b_2|) \leq C$. Then it is decidable whether or not (E_2) is solvable.

Proof.

(I). Consider all solutions (α,β) such that

$$|b_1g_1(\alpha)b_2| \geq |a_1f_1(\alpha)a_2| \quad \ldots\ldots (\$_1)$$

where the assumption of the lemma is satisfied, that is,

$$|b_1g_1(\alpha)b_2| - |a_1f_1(\alpha)a_2| \leq C.$$

Thus in this part (I) of our proof, whenever we say "consider a solution (α,β) of (E_2) (all solutions (α,β) of (E_2))" we mean a solution (α,β) of (E_2) (all solutions (α,β) of (E_2)) satisfying restriction $(\$_1)$.

Let m be a nonnegative integer, $m \le C$. (Intuitively speaking, for each $m \le C$ we will consider a possibility that (E_2) has a solution (α,β) where $|b_1 g_1(\alpha) b_2| - |a_1 f_1(\alpha) a_2| = m$.) Let r be the minimal positive integer such that $r|f_2(0)| > m$ and let s be the minimal positive integer such that $s|g_1(0)| > m$.

Let $w = (pref_m(f_2(0^r)) \setminus f_2(0^r)$, $z = g_2(0^r)$

$\hat{w} = f_1(0^s)a_2\, pref_m(f_2(0^r))$ and $\hat{z} = g_1(0^s)b_2$.

Consider the following system of equations

$$\left.\begin{aligned} a_1 f_1(x)\,\hat{w} &= b_1 g_1(x)\,\hat{z} \\ w\, f_2(y)\, a_3 &= z\, g_2(y)\, b_3 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (F_m)$$

Let R be the maximal among all r (constructed for all $m \le C$) and let S be the maximal among all s (constructed for all $m \le C$).

(i). Notice that if (F_m) has a solution then also (E_2) has a solution. Moreover, given a solution of (F_m) one can effectively construct a solution of (E_2).

This is seen as follows.

Assume that (α,β) is a solution of (F_m).

Then

$a_1 f_1(\alpha)\hat{w}\, w\, f_2(y) a_3 =$

$= a_1 f_1(\alpha) f_1(0^s) a_2 f_2(0^r) f_2(\beta) a_3 =$

$= b_1 g_1(\alpha) g_1(0^s) b_2 g_2(0^r) g_2(\beta) b_3$

and consequently $(\alpha\, 0^s, 0^r\, \beta)$ is a solution of (E_2).

Thus (i) holds.

Next for every $\gamma \in \{0\}^+$ such that $|\gamma| \le S$ consider the equation

$$a_1 f_1(\gamma) a_2 f_2(y) a_3 = b_1 g_1(\gamma) b_2 g_2(y) b_3 \quad \cdots\cdots\cdots\cdots \quad (T_\gamma)$$

and for every $\gamma \in \{0\}^+$ such that $|\gamma| \leq R$ consider the equation

$$a_1f_1(x)a_2f_2(\gamma)a_3 = b_1g_1(x)b_2g_2(\gamma)b_3 \quad \ldots\ldots (U_\gamma)$$

(ii). Notice that if (T_γ) has a solution then also (E_2) has a solution. Moreover, given a solution of (T_γ) one can effectively construct a solution of (E_2).

This follows because, obviously, if β is a solution of (T_γ) then (γ,β) is a solution of (E_2).

(iii). Analogously we see that if (U_γ) has a solution then (E_2) has a solution and moreover, given a solution of (U_γ) one can effectively construct a solution of (E_2).

(iv). Consider now possible solutions (α,β) of (E_2).

(i.v.1). If $|\alpha| > S$ and $|\beta| > R$ then, obviously, for some $m \leq C$ the system of equations (F_m) has a solution.

(i.v.2). If $|\alpha| \leq S$ then, obviously, there exists a γ such that (T_γ) has a solution.

(i.v.3). If $|\beta| \leq R$ then, obviously, there exists a γ such that (U_γ) has a solution.

Now the lemma (under the condition $(\$_1)$) follows from (i) through (iv), Theorem 1 and the fact that each system (F_m) of equations consists of two independent equations.

(II). Consider all solutions (α,β) such that

$$|a_1f_1(\alpha)a_2| \geq |b_1g_1(\alpha)b_2| \quad \ldots\ldots (\$_2)$$

where the assumption of the lemma is satisfied.

Analogously to the case (I) we can prove that it is decidable whether or not there exists a solution (α,β) of (E_2) such that both $(\$_2)$ and the assumption of the lemma is satisfied. Moreover, if the answer is positive one can effectively find such a solution.

Cases (I) and (II) together yield the proof of Lemma 2. □

Theorem 2. It is decidable whether or not an arbitrary unary 2-fold equation has a solution.

Proof.

Consider the equation (E_2).

We investigate separately the following cases.

[I]. There exists a pair of homomorphisms h_1, h_2 from the set $\{f_1, f_2, g_1, g_2\}$ such that h_1 *dagr* h_2.

We distinguish the following subcases of this case.

[I.1]. f_1 *dagr* g_1.

Consider all possible words γ, δ of the form $\gamma = a_1 f_1(x)$, $\delta = b_1 g_1(x)$ such that $x \neq \Lambda$ and either γ *pref* δ or δ *pref* γ. By Lemma 1 the set Z of all such pairs is finite and can be effectively constructed. Now for each x such that $(\gamma = a_1 f_1(x), \delta = b_1 g_1(x)) \in Z$ we consider the equation

$$\gamma a_2 f_2(y) a_3 = \delta b_2 g_2(y) b_3 \ldots\ldots (E_{1,x}).$$

Clearly (α, β) is a solution of (E_2) if and only if β is a solution of $(E_{1,\alpha})$. Since the number of equations $(E_{1,x})$ is finite and all of them can be effectively constructed, it follows from Theorem 1 that Theorem 2 holds in the case [I.1].

[I.2]. f_2 *dagr* g_2.

This case is analogous to the case [I.1].

[I.3]. f_1 *dagr* f_2, f_1 *agr* g_1 and f_2 *agr* g_2.

Notice that (because *agr* is an equivalence relation) we have that f_2 *dagr* g_1 and f_1 *dagr* g_2. We will demonstrate now the following.

(i). If (α,β) is a solution of (E_2) then there exists effectively a positive integer C such that $abs(|a_1f_1(\alpha)a_2| - |b_1g_1(\alpha)b_2|) \leq C$.

To prove this claim we consider separately two cases.

(i.1). Let $|b_1g_1(\alpha)b_2| \geq |a_1f_1(\alpha)a_2|$.

Let $D = \max\{2, 2\frac{|f_2(0)|}{|g_1(0)|}\}$ and $C = D\max\{|f_2(0)|, |g_1(0)|\} + abs(|a_3|-|b_3|) + |b_2|$.

Assume that $|b_1g_1(\alpha)b_2| - |a_1f_1(\alpha)a_2| > C$.

Then there exist words $\gamma, \delta \in \Sigma^*$ such that $g_1(\gamma)$ *sub* $f_2(\delta)$ where $|\gamma| > D$, which contradicts Lemma 1.

Hence $|b_1g_1(\alpha)b_2| - |a_1f_1(\alpha)a_2| \leq C$.

(i.2). Let $|a_1f_1(\alpha)a_2| \geq |b_1g_1(\alpha)b_2|$.

Let $D = \max\{2, 2\frac{|f_1(0)|}{|g_2(0)|}\}$ and $C = D\max\{|f_1(0)|, |g_2(0)|\} + abs(|a_3| - |b_3|) + |a_2|$.

Assume that $|a_1f_1(\alpha)a_2| - |b_1g_1(\alpha)b_2| > C$.

Then there exist words $\gamma, \delta \in \Sigma^*$ such that $g_2(\gamma)$ *sub* $f_1(\delta)$ where $|\gamma| > D$, which contradicts Lemma 1.

Hence $|a_1f_1(\alpha)a_2| - |b_1g_1(\alpha)b_2| \leq C$.

Now, (i) follows from (i.1) and (i.2).

Thus Lemma 2 implies that the theorem holds also in this case.

[II]. Every pair of homomorphisms h_1, h_2 from the set $\{f_1, f_2, g_1, g_2\}$ is such that h_1 *agr* h_2.

Let p be a word of the smallest length such that

$f_1(0) = p^{k_1}$, $f_2(0) = p_1^{k_2}$, $g_1(0) = p_2^{\ell_1}$ and $g_2(0) = p_3^{\ell_2}$

where all of the words p, p_1, p_2, p_3 are pairwise cyclic conjugates.

First of all we notice that by Theorem 1 it is decidable whether or not (E_2) has a solution (α,β) such that either $|\alpha| \leq 2$ or $|\beta| \leq 2$ (and if such a solution exists it can be effectively found).

Thus from now on we assume that x, y in (E_2) are such that both $|x| > 2$ and $|y| > 2$.

We construct now the 2-fold equation

$$\bar{a}_1\bar{f}_1(x)\bar{a}_2\bar{f}_2(y)\bar{a}_3 = \bar{b}_1\bar{g}_1(x)\bar{b}_2\bar{g}_2(y)\bar{b}_3 \quad \ldots\ldots\ldots (\bar{E}_2)$$

where

$\bar{f}_1(0) = p^{k_1}$, $\bar{f}_2(0) = p^{k_2}$, $\bar{g}_1(0) = p^{\ell_1}$, $\bar{g}_2(0) = p^{\ell_2}$,

$\bar{a}_1 = a_1 p p^{k_1 - 1}$,

$\bar{a}_2 = a_2 p_1' p^{k_2 - 1}$ where $p_1 = p_1' p_1''$ and $p_1'' p_1' = p$,

$\bar{a}_3 = p_1'' a_3$,

$\bar{b}_1 = b_1 p_2' p^{\ell_1 - 1}$ where $p_2 = p_2' p_2''$ and $p_2'' p_2' = p$,

$\bar{b}_2 = p_2'' b_2 p_3' p^{\ell_2 - 1}$ where $p_3 = p_3' p_3''$ and $p_3'' p_3' = p$, and

$\bar{b}_3 = p_3'' b_3$.

From the construction of (E_2) it easily follows that (E_2) has a solution if and only if $(\bar{E}_2)$ has a solution and moreover, given a solution of $(\bar{E}_2)$ one can construct effectively a solution of (E_2).

We consider now the following cases.

(i). $\bar{a}_2$ is not a power of p.

Let $q = \max\{|f_1(0)|, |f_2(0)|, |g_1(0)|, |g_2(0)|\}$, $w = \bar{a}_1\bar{a}_2\bar{a}_3\bar{b}_1\bar{b}_2\bar{b}_3$ and let $C = 3q(|w| + |p|)$.

Let (α,β) be a solution of $(\bar{E}_2)$.

Then we have

$abs(|\bar{a}_1\bar{f}_1(\alpha)\bar{a}_2| - |\bar{b}_1\bar{g}_1(\alpha)b_2|) \leq C$ ....($\$_3$)

which is seen as follows.

Assume that $|\bar{b}_1\bar{g}_1(\alpha)\bar{b}_2| - |\bar{a}_1\bar{f}_1(\alpha)\bar{a}_2| > C$. Then the choice of C guarantees that $|\bar{a}_1\bar{f}_1(\alpha)\bar{a}_2| - |\bar{b}_1| > |\bar{a}_2| + |p|$ and $|\bar{a}_2\bar{f}_2(\beta)\bar{a}_3| - |\bar{b}_3| > |\bar{a}_2| + |p|$. Consequently $p\bar{a}_2p$ is a subword of a word of the form p^s for some $s \geq 3$. Since p was a word of the smallest length such that $f_1(0)$ is a power of p, this implies that $\bar{a}_2$ is a power of p; a contradiction.

Similarily we get a contradiction when we assume that $|\bar{a}_1\bar{f}_1(\alpha)\bar{a}_2| - |\bar{b}_1\bar{g}_1(\alpha)\bar{b}_2| > C$.

Consequently ($\$_3$) holds.

(ii). Analogously, assuming that $\bar{b}_2$ is not a power of p and (α,β) is a solution of $(\bar{E})$, we get that ($\$_3$) holds.

Hence we have the following.

(iii). If either $\bar{a}_2$ is not a power of p or $\bar{b}_2$ is not a power of p then one can decide whether or not $(\bar{E}_2)$ has a solution, and if $(\bar{E}_2)$ is solvable then one can effectively find a solution of it.

To conclude the proof assume now that

(iv) both $\bar{a}_2$ and $\bar{b}_2$ are powers of p, say $\bar{a}_2 = p^s$ and $\bar{b}_2 = p^t$.

First of all notice that by Theorem 1 it is decidable whether or not $(\bar{E}_2)$ has a solution (α,β) such that either $|\alpha| \leq |\bar{a}_1\bar{a}_2\bar{a}_3\bar{b}_1\bar{b}_2\bar{b}_3|$ or $|\beta| \leq |\bar{a}_1\bar{a}_2\bar{a}_3\bar{b}_1\bar{b}_2\bar{b}_3|$; moreover, if such a solution exists one can effectively find one.

Thus we assume that both α and β are longer than $|\bar{a}_1\bar{a}_2\bar{a}_3\bar{b}_1\bar{b}_2\bar{b}_3|$.

We have to consider now four cases depending on whether or not the values of $(|\bar{a}_1| - |\bar{b}_1|)$ and $(|\bar{a}_3| - |\bar{b}_3|)$ are negative or not.

(iv.1). Assume that $|\bar{a}_1| \leq |\bar{b}_1|$ and $|\bar{a}_3| \geq |\bar{b}_3|$ (the reader can check the remaining cases).

First of all we notice that both $|\bar{b}_1| - |\bar{a}_1|$ and $|\bar{a}_3| - |\bar{b}_3|$ are divisible by $|p|$; since otherwise p could not be a minimal period of $f_1(0)$. Say $|\bar{b}_1| - |\bar{a}_1| = r\,|p|$ and $|\bar{a}_3| - |\bar{b}_3| = u\,|p|$ for some $r, u \geq 0$.

Consequently $(\bar{E}_2)$ has a solution if and only if the equation

$r + k_1 n + s + k_2 m = \ell_1 n + t + \ell_2 m + u$ ....(G)

in variables n and m has a positive solution (that is we assume both n and $m > 0$). Moreover, if (n,m) is a positive solution of G then $(0^n, 0^m)$ is a solution of $(\bar{E}_2)$.

Consequently the theorem holds also in the case (iv.1).

The remaining three subcases of (iv) are proved in the same case.

But (i) through (iv) conclude the proof of case [II] and the theorem follows from [I] and [II].

Remark. In this paper we have demonstrated the decidability of the unary 2-fold solvability problem. Our method of solution consists of showing "directly" how to solve the problem in some cases and how to reduce the problem to the unary 1-fold solvability problem in the remaining cases. This method can be elaborated to prove the decidability of the general unary n-fold solvability problem. However, the solution becomes quite tedious and for this reason we have decided to present a solution of the problem when n = 2. The interested reader can certainly carry out the obvious generalization himself.

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