FINDING A HOMOMORPHISM BETWEEN TWO WORDS IS NP-COMPLETE

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Abstract. We demonstrate that to find a homomorphism between two words x and y where letters of x can be chosen from an infinite alphabet and y is a word over a two letter alphabet is an NP-complete problem.

Looking for NP-complete problems today forms an active research area within complexity theory (see, e.g., [1]). Showing that a problem is NP-complete often contributes to our understanding of the difficulty of the problem and of the nature of NP-complete problems in general.

In this note we demonstrate that a very common problem from formal language theory is NP-complete: finding a nonerasing homomorphism between two words x and y (providing we do not restrict a priori the size of the alphabet of the word x; y can be chosen over a two-letter alphabet.) The problem remains NP-complete even when arbitrary homomorphisms are admitted. It may very well be one of the simplest NP-complete problems known. On the other hand one can easily see that if the size of the alphabet is limited a' priori then the problem is in *P*.

Formally it is defined as follows. (In the sequel, given a word x, *alph* x denotes the set of letters appearing in x and $\#_a x$ denotes the number of occurrences of the letter a in x; $HOM(\Sigma_1,\Sigma_2)$ denotes the class of nonerasing homomorphisms from Σ_1^* into Σ_2^*).

Let Σ be an infinite alphabet, Δ its subset containing two elements, $\Delta = \{b,¢\}$ say. Let

$$MATCH(\Sigma,\Delta) = \{(x,y) : \mathit{alph}\, x \subset \Sigma,\ \mathit{alph}\, y \subseteq \Delta \text{ and there exists a homomorphism } h \text{ in } HOM(\mathit{alph}\, x, \mathit{alph}\, y) \text{ such that } h(x) = y\}.$$

Theorem. Membership in $MATCH(\Sigma,\Delta)$ is NP-complete.

Proof.

(i) Obviously membership in $MATCH(\Sigma,\Delta)$ is in NP.

(ii) Since 3-satisfiability is NP-complete (see, e.g., [1]) it suffices to show that for every Boolean expression Ψ in 3-conjunctive normal form (3-CNF) there exist a word x_Ψ in Σ^+ and y_Ψ in Δ^+ such that

(*). . .$(x_\Psi,y_\Psi) \in MATCH(\Sigma,\Delta)$ if and only if Ψ is satisfiable.

To this aim we proceed as follows.

Let $V = \{P_1,\ldots,P_\ell\}$ be a set of Boolean variables and let

$\Psi = (P'_{i_1} \vee P'_{j_1} \vee P'_{k_1}) \wedge \ldots \wedge (P'_{i_n} \vee P'_{j_n} \vee P'_{k_n})$,

with $i_q, j_q, k_q \in \{1,\ldots,\ell\}$ be a Boolean expression over V in 3-CNF where, for $q \in \{1,\ldots,n\}$, each P'_{i_q} (P'_{j_q}, P'_{k_q} respectively) is either a variable P_{i_q} (P_{j_q}, P_{k_q} respectively) or its negation $\overline{P}_{i_q}$ ($\overline{P}_{j_q}, \overline{P}_{k_q}$ respectively).

Our construction of words x_Ψ, y_Ψ takes two steps.

STEP 1.

We will construct a finite set W of pairs of words (α,β) with $\alpha \in V_\Psi^+, V_\Psi \subset \Sigma$, and $\beta \in \Delta^+$ such that

$$(**) \ldots \begin{cases} \Psi \text{ is satisfiable if and only if there exists a homomorphism} \\ h \text{ in HOM}(V_\Psi,\{b\}) \text{ such that, for every } (\alpha,\beta) \text{ in } W,\ h(\alpha) = \beta. \end{cases}$$

Let $\overline{V} = \{\overline{P}_q : 1 \le q \le \ell\}$. Clearly we can assume that $V \cup \overline{V} \subseteq \Sigma$.

Let $T_1,\ldots,T_n$, $U_1,\ldots,U_n$ be new elements of Σ different from each other.

Let $V_\Psi = V \cup \overline{V} \cup \{T_q : 1 \le q \le n\} \cup \{U_q : 1 \le q \le n\}$ and let

$W_1 = \{(P_q\overline{P}_q, b^3) : 1 \le q \le \ell\}$,

$W_2 = \{(T_q P'_{i_q} P'_{j_q} P'_{k_q}, b^6) : 1 \le q \le n\}$

$W_3 = \{(T_q U_q, b^4) : 1 \le q \le n\}$.

Let $W = W_1 \cup W_2 \cup W_3$.

We prove (**) as follows.

(1) Assume that there exists a homomorphism h in $\text{HOM}(V_\Psi,\{b\})$ such that, for every (α,β) in W, $h(\alpha) = \beta$.

Then let f be the valuation of V such that for every $q \in \{1,\ldots,\ell\}$, $f(P_q) =$ *false* if and only if $h(P_q) = b^2$. Since $W_1 \subseteq W$, f is well defined (and it follows that $f(P_q) =$ *true* if and only if $h(P_q) = b$).

Since $W_3 \subseteq W$, $1 \le |h(T_q)| \le 3$ for $1 \le q \le n$ and, because $W_2 \subseteq W$, this implies that $3 \le |h(P'_{i_q} P'_{j_q} P'_{k_q})| \le 5$ for $1 \le q \le n$. Thus for every $q \in \{1,\ldots,n\}$ either $|h(P'_{i_q})| = 1$ or $|h(P'_{j_q})| = 1$ or $|h(P'_{k_q})| = 1$ which implies that for every $q \in \{1,\ldots,n\}$, $f((P'_{i_q} \vee P'_{j_q} \vee P'_{k_q})) =$ *true* and so Ψ is satisfied by f.

(2) Let Ψ be satisfiable and let f be a valuation of V which satisfies Ψ. Let h be the homomorphism on V_Ψ^* defined by: for $1 \le q \le \ell$,

if $f(P_q) =$ *true* then $h(P_q) = b$ and $h(\overline{P}_q) = b^2$,

if $f(P_q) =$ *false* then $h(P_q) = b^2$ and $h(\overline{P}_q) = b$,

for $1 \le q \le n$,

if all three of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b^3$ and $h(U_q) = b$,

if only two of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b^2$ and $h(U_q) = b^2$,

if only one of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ is equal *true*

then $h(T_q) = b$ and $h(U_q) = b^3$.

It follows directly from the definition of h that indeed for every (α,β) in W, $h(\alpha) = \beta$.

Thus (**) holds.

STEP 2.

Now given W from STEP 1 we construct x_Ψ, y_Ψ satisfying (*) as follows Let $W = \{(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\}$ for some $m \ge 3$ and let $x_\Psi = ¢\, \alpha_1\, ¢\ldots¢\, \alpha_m$ and $y_\Psi = ¢\, \beta_1\, ¢\ldots¢\, \beta_m$.

Note that if h is a homomorphism from *alph* $x_\Psi = V_\Psi \cup \{¢\}$ into *alph* $y_\Psi = B$ such that $h(x_\Psi) = y_\Psi$ then $h(¢) = ¢$ (because both x_Ψ and y_Ψ

start with ¢ and $¢ \notin$ *alph* β_1). Since $\#_{¢}x_{\Psi} = \#_{¢}y_{\Psi}$ this implies that

(***) . . . $\left\{\right.$ there exists a homomorphism h from *alph* x_{Ψ} into *alph* y_{Ψ} such that $h(x_{\Psi}) = y_{\Psi}$ if and only if there exists a homomorphism g from V_{Ψ} into {b} such that $g(\alpha_q) = \beta_q$ for $1 \leq q \leq m$.

But (***) and (**) imply (*) and so the theorem holds.

Remark. If we change the definition of MATCH(Σ,Δ) to the definition of $\text{MATCH}_{\Lambda}(\Sigma,\Delta)$ by allowing arbitrary rather than only nonerasing homomorphisms then the theorem still remains true. That is we get the result: "The membership in $\text{MATCH}_{\Lambda}(\Sigma,\Delta)$ is NP-complete." The main idea of the proof is the same and the only changes to be made are the following ones:

(1) For every (α,β) in W_1 set $\beta = b$ (rather than $\beta = b^3$).

(2) For every (α,β) in W_2 set $\beta = b^3$ (rather than $\beta = b^6$)

(3) For every (α,β) in W_3 set $\beta = b^2$ (rather than $\beta = b^4$)

(4) Given a homomorphism h "satisfying" W set the valuation f of V by: $f(P_q) =$ *true* if and only if $h(P_q) = b$.

(5) Given a valuation f of V satisfying Ψ set the homomorphism h by

if $f(P_q) =$ *true* then $h(P_q) = b$ and $h(\overline{P}\ ) = \Lambda$,

if $f(P_q) =$ *false* then $h(P_q) = \Lambda$ and $h(\overline{P}_q) = b$,

if all three of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = \Lambda$ and $h(U_q) = b^2$,

if only two of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b$ and $h(U_q) = b$,

if only two of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b^2$ and $h(U_q) = \Lambda$.

(6) In *STEP 2* define

$x_\Psi = ¢\ ¢\ \alpha_1\ ¢\ldots¢\ \alpha_m\ ¢\ \alpha_1\ ¢\ldots¢\ \alpha_m$ and

$y_\Psi = ¢\ ¢\ \beta_1\ ¢\ldots¢\ \beta_m\ ¢\ \beta_1\ ¢\ldots¢\ \beta_m$.

REFERENCES

[1] A. Aho, J. Hopcroft and J. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass., 1974.

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