All correspondence to second author

EVERY TWO EQUIVALENT DOL SYSTEMS HAVE A REGULAR TRUE ENVELOPE

by

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ABSTRACT

Extending some proof techniques from [2] and [3] we solve an open problem from [1] and prove that every two equivalent DOL systems have a regular true envelope.

0. INTRODUCTION

This paper pursues further the research started in [2] and [3]. Its aim is to demonstrate further the usefulness of elementary homomorphisms and proof techniques around them for solving (using "systematic" proof techniques) various problems concerning DOL systems. In particular we solve an open problem from [1] and prove that every two equivalent DOL systems have a true regular envelope.

We assume the reader to be familiar with the rudiments of automata theory and the rudiments of the theory of DOL systems.

I. PRELIMINARIES

Throughout the paper we use standard language theoretic terminology. Perhaps the following notation requires some explanation.

(i) If K is a finite set, then #K denotes its cardinality. For an integer x , $\|x\|$ denotes its absolute value.

(ii) For a word α , $|\alpha|$ denotes its length and Sub(α) denotes the set of all subwords of α. For a language K, SubK = $\bigcup_{\alpha \in K}$ Sub(α).

(iii) HOM(Σ,Δ) denotes the set of all homomorphisms from Σ^* into Δ^*. A composition of homomorphisms $h_1,\ldots,h_k$ is written $h_k \ldots h_1$. For a homomorphism h, maxr(h) = $\max\{ |\alpha| : (\exists a)_\Sigma [h(a) = \alpha] \}$. For a set of homomorphisms H, Sem(H) denotes the semigroup generated by H.

(iv) Let A = $(\Sigma,Q,\delta,q_{in},F)$ be a finite automaton. For q in Q and γ in Σ^*, trace(q,γ) is the sequence of states encountered when starting at q and following transitions forced by γ. A sequence of states $q,q_1,\ldots,q_n,q$ where $q_i \neq q_j$ for $1 \leq i \neq j \leq n$ is called a simple loop (in Q).

Next we introduce some terminology and notation which is useful when dealing with homomorphisms.

Definition 1. Let h,g $\in$ HOM(Σ,Δ).

1) A language $K \subseteq \Sigma^*$ is an identifying set for h,g if h(α) = g(α) for every α in K. We say then that h equals g on K and write $h =_K g$. Similarly if $\tau = \omega_0,\omega_1,\ldots$ is a sequence of words, we say that h equals g on τ if $h =_{K_\tau} g$ where K_τ is the set of all words that occur in the sequence τ, we write then $h =_\tau g$.

2) A language $K \subseteq \Sigma^*$ is the maximal identifying set for h,g, denoted as MID(h,g), if $K = \{ \alpha \in \Sigma^* : h(\alpha) = g(\alpha) \}$.

3) If K is an identifying set for h,g and there exists a constant C such that, for every prefix (respectively subword) α of K, we have

$\| |h(\alpha)| - |g(\alpha)| \| < C$ then we say that h,g are prefix balanced (respectively subword balanced) on K. Since it follows directly from the definition that h,g are prefix balanced on K if and only if h,g are subword balanced on K, we will simply say that h,g are balanced on K.

4) The balanced family of h,g, denoted as BAL(h,g), is the family consisting of all these subsets of MID(h,g) on which h and g are balanced. For Z in BAL(h,g), b(Z,h,g) denotes the minimal constant C such that $\| |h(\alpha)| - |g(\alpha)| \| < C$ for all subwords α of Z.

5) Let α in Σ^* be such that $h(\alpha)$ is a prefix of $g(\alpha)$ or $g(\alpha)$ is a prefix of $h(\alpha)$, in particular it can be that $h(\alpha) = g(\alpha)$. Then the delay of h,g on α, denoted as $\underline{del}_{h,g}(\alpha)$ is the word over Δ such that either $h(\alpha)\underline{del}_{h,g}(\alpha) = g(\alpha)$ or $g(\alpha)\underline{del}_{h,g}(\alpha) = h(\alpha)$.

6) Let k be a nonnegative integer. The k-delayed identifying set for h,g, denoted as $MID_k(h,g)$, is defined by

$$MID_k(h,g) = \{ \alpha \in \Sigma^* : h(\alpha) = g(\alpha) \text{ and } (\forall\beta)_{\underline{Sub}(\alpha)}[\| |h(\beta)| - |g(\beta)| \| \leq k] \} .$$

The following result follows directly from the definition.

Lemma 1. Let h,g be homomorphisms on Σ^* and let $\alpha,\bar{\alpha},\beta \in \Sigma^*$. If $\underline{del}_{h,g}(\alpha) = \underline{del}_{h,g}(\bar{\alpha})$ then $\underline{del}_{h,g}(\alpha\beta) = \underline{del}_{h,g}(\bar{\alpha}\beta)$.

It is easy to see that given h,g and k there exists a finite automaton which accepts $MID_k(h,g)$. One simply keeps an information about the "current delay between h and g" in a state; since the length of such a delay cannot exceed k a finite number of states suffices. We leave the formal proof of this result to the reader.

Theorem 1. $MID_k(h,g)$ is a regular set for arbitrary homomorphisms h,g and an arbitrary nonnegative integer k.

Finally let us recall from [1] the notion of a (regular) true envelope for a pair of DOL systems.

Definition 2. Let $G_1 = (\Sigma, h_1, \omega)$ and $G_2 = (\Sigma, h_2, \omega)$ be DOL systems. A language K is a true envelope for G_1, G_2 if

(i) $L(G_1) \subseteq K$ and $L(G_2) \subseteq K$,

(ii) $K \subseteq MID(h_1, h_2)$.

K is called a regular true envelope for G_1, G_2 if K is a regular language and a true envelope for G_1, G_2.

II. BALANCED HOMOMORPHISMS

In this section we investigate several situations in which a pair of homomorphisms is balanced on a language.

We start by providing necessary and sufficient conditions for equality of two homomorphisms on a regular language[1]. First, we need some auxilliary notation.

Let $A = (\Sigma, Q, \delta, q_{in}, F)$ be a deterministic finite automaton and let $p \in Q$. Then

$INIT(p) = \{ \alpha : \delta(q_{in},\alpha) = p \}$,

$FIN(p) = \{ \beta : \delta(p,\beta) \in F \}$,

$Loop(p) = \{ \gamma : \delta(p,\gamma) = p$ and $\underline{trace}(p,\gamma)$ is a simple loop $\}$,

$Z_0(p) = \{ \alpha : \delta(q_{in},\alpha) = p$ and $\underline{trace}(p,\gamma)$ contains no loop $\}$,

for every $k \geq 1$

$Z_k(p) = \{ \alpha : \delta(q_{in},\alpha) = p$ and $\underline{trace}(p,\gamma)$ contains no loops except for no more than k simple loops $\}$.

Theorem 2. Let $A = (\Sigma, Q, \delta, q_{in}, F)$ be a deterministic finite state automaton and let h,g be two homomorphisms on Σ^*. Then $h =_{T(A)} g$ if and only if

1. $(\forall p)_Q (\forall \alpha)_{Z_0(p)} [\ h(\alpha) = g(\alpha)$ if and only if $p \in F\ ]$

2. There exists a function $\psi_{Z_0} : Q \rightarrow \Sigma^*$ such that,

for every q in Q,

either 2.1. $(\ (\forall \alpha)_{Z_0(p)} [\ h(\alpha)\psi_{Z_0}(p) = g(\alpha)\ ]\ )$ and

$(\ (\forall \gamma)_{Loop(p)} [\ \psi_{Z_0}(p) g(\gamma) = h(\gamma)\psi_{Z_0}(p)\ ]\ )$,

or 2.2. $(\ (\forall \alpha)_{Z_0(p)} [\ g(\alpha)\psi_{Z_0}(p) = h(\alpha)\ ]\ )$ and

$(\ (\forall \gamma)_{Loop(p)} [\ \psi_{Z_0}(p) h(\gamma) = g(\gamma)\psi_{Z_0}(p)\ ]\ )$.

I. "If" part:

(i) First we will show that the condition 2^o from the statement of the Theorem remains valid when, for an arbitrary $k \geq 0$, we replace Z_0 by Z_k and moreover $\psi_{Z_k} = \psi_{Z_0}$.

We show it by induction on k.

k = 0. It is true by the assumptions of the Theorem.

Assume that the statement is true when replacing Z_0 by Z_k (referred to as the "k-modified statement 2") and let us replace Z_0 by Z_{k+1}.

Let $p \in Q$.

(i.1) Assume that p satisfies the condition 2.1 of the k-modified statement 2.

Take $\alpha \in Z_{k+1}(p) \backslash Z_k(p)$.

Then $\alpha = \bar{\alpha}\bar{\gamma}\bar{\beta}$ where there exists a state r in Q such that $\bar{\alpha} \in Z_j(r)$, $\gamma \in \mathrm{Loop}(r)$ and $\bar{\alpha}\bar{\beta} \in Z_\ell(p)$ for some $j, \ell \leq k$.

(i.1.1) Assume that r satisfies the condition 2.1 of the modified statement 2.

Then $h(\bar{\alpha})\psi_{Z_0}(r) = g(\bar{\alpha})$ and $h(\bar{\alpha}\bar{\gamma})\psi_{Z_0}(r) = g(\bar{\alpha}\bar{\gamma})$.

But $\delta(r,\bar{\beta}) = p$ and, because $p = \delta(q_{in},\bar{\alpha}\bar{\gamma}\bar{\beta}) = \delta(q_{in},\bar{\alpha}\bar{\beta})$, Lemma 1 implies that $h(\bar{\alpha}\bar{\gamma}\bar{\beta})\psi_{Z_0}(p) = g(\bar{\alpha}\bar{\gamma}\bar{\beta})$.

(i.1.2) Assume that r satisfies the condition 2.2 of the modified statement 2.

Then $g(\bar{\alpha})\psi_{Z_0}(r) = h(\bar{\alpha})$ and $g(\bar{\alpha}\bar{\gamma})\psi_{Z_0}(r) = h(\bar{\alpha}\bar{\gamma})$.

But $\delta(r,\bar{\beta}) = p$ and, because $p = \delta(q_{in},\bar{\alpha}\bar{\gamma}\bar{\beta}) = \delta(q_{in},\bar{\alpha}\bar{\beta})$, Lemma 1 implies that $h(\bar{\alpha}\bar{\gamma}\bar{\beta})\psi_{Z_0}(p) = g(\bar{\alpha}\bar{\gamma}\bar{\beta})$.

(i.1.3) Hence if p satisfies the condition 2.1 of the k-modified statement 2 then it also satisfies the condition 2.1 of the (k + 1)-modified statement 2.

(i.2) Similarly we prove that if p satisfies the condition 2.2 of the k-modified statement 2 then it also satisfies the condition 2.2 of the (k + 1)-modified statement 2.

This completes our inductive proof.

(ii) From (i) it clearly follows that, for every p in Q,

<u>either</u> $((\forall\alpha)_{INIT(p)}[\ h(\alpha)\psi_{Z_0}(p) = g(\alpha)\ ])$ and

$((\forall\gamma)_{Loop(p)}[\ \psi_{Z_0}(p)g(\gamma) = h(\gamma)\psi_{Z_0}(p)\ ])$

<u>or</u> $((\forall\alpha)_{INIT(p)}[\ g(\alpha)\psi_{Z_0}(p) = h(\alpha)\ ])$ and

$((\forall\gamma)_{Loop(p)}[\ g(\gamma)\psi_{Z_0}(p) = \psi_{Z_0(p)}h(\gamma)\ ])$.

But this together with the condition 1 of the statement of the theorem implies that $(\forall p)_Q(\forall\alpha)_{INIT(p)}[\ h(\alpha) = g(\alpha)$ if and only if $p \in F\ ]$ which in turns implies that $h =_{T(A)} g$.

II. "Only if" part:

Let us assume that $h =_{T(A)} g$.

This clearly implies condition 1 of the statement of the theorem.

Let $p \in Q$ and let $\alpha \in Z_0(p)$.

(i) Assume that ζ is such that $h(\alpha)\zeta = g(\alpha)$.

Let $\beta \in FIN(p)$. Then $h(\alpha)h(\beta) = g(\alpha)g(\beta) = h(\alpha)\zeta g(\beta)$ and consequently $h(\beta) = \zeta g(\beta)$.

(i.1) Let $\bar{\alpha} \in Z_0(p)$ where $\bar{\alpha} \neq \alpha$.

Then $g(\bar{\alpha})g(\beta) = h(\bar{\alpha})h(\beta) = h(\bar{\alpha})\zeta g(\beta)$ and so $h(\bar{\alpha})\zeta = g(\bar{\alpha})$.

(i.2) Let $\gamma \in Loop(p)$.

Then $h(\alpha)h(\gamma)h(\beta) = h(\alpha)h(\gamma)\zeta g(\beta)$ and

$g(\alpha)g(\gamma)g(\beta) = h(\alpha)\zeta g(\gamma)g(\beta)$.

Since $h(\alpha)h(\gamma)h(\beta) = g(\alpha)g(\gamma)g(\beta)$, we get $h(\gamma)\zeta = \zeta g(\gamma)$.

(i.3) From (i.1) and (i.2) it follows that if we set $\zeta = \psi_{Z_0}(p)$ then condition 2.1 of the statement of the theorem holds.

(ii) Similarly if we assume that ζ is such that $h(\alpha) = g(\alpha)\zeta$, we can show that the condition 2.2 of the statement of the theorem holds if we set $\zeta = \psi_{Z_0}(p)$.

Corollary 1. Let $h =_K g$ where K is a regular language. Then $K \in BAL(h,g)$.

Proof.

Let $K = T(A)$ where $A = (\Sigma, Q, \delta, q_{in}, F)$ is a deterministic finite automaton and let $C_K = 2 \cdot \max \{ |\psi_{Z_0}(p)| : p \in Q \}$ where ψ_{Z_0} is defined as in the statement of Theorem 2. Let $\alpha \in \underline{Sub}(K)$. Thus there exist a word ζ in K such that $\zeta = \beta\alpha\gamma$. Let $p = \delta(q_{in}, \beta)$ and $r = \delta(q_{in}, \beta\alpha)$. Then, see (ii) from the proof of "if" part of Theorem 2,
$\| |h(\alpha)| - |g(\alpha)| \| < |\psi_{Z_0}(p)| + |\psi_{Z_0}(r)|$ and so the Corollary holds.

The following result says essentially that the property of being balanced carries over through closures of homomorphic diagrams; the result that is needed very much for the proof of our main theorem.

Theorem 3. Let $f_1, g_1, h_1, f_2, g_2, h_2$ be homomorphisms and let Z be a language such that

(1) f_1 and f_2 are Λ-free,

(2) $h_1 f_1 = g_1$ and $h_2 f_2 = g_2$, and

(3) $Z \in (B(f_1, f_2) \cap B(g_1, g_2))$.

Then $f_1(Z) = f_2(Z) \in B(h_1, h_2)$.

Proof.

We start with the following observation.

(i) Let ϕ_1, ϕ_2 be homomorphisms, $X \in B(\phi_1,\phi_2)$ and let $b(X,\phi_1,\phi_2) = n$. Let $\beta \in X$ and let $\gamma = \phi_1(\beta) = \phi_2(\beta)$. Let Q be an occurrence in β and let L_k, L_ℓ, R_i, R_j for $k,\ell,i,j \in \{1,2\}$, $k \neq \ell$, $i \neq j$, be occurrences in γ such that

L_k is the leftmost occurrence in γ that is derived by ϕ_k

L_ℓ is the leftmost occurrence in γ that is derived by ϕ_ℓ from Q,

R_i is the rightmost occurrence in γ that is derived by ϕ_i from Q, and

R_j is the rightmost occurrence in γ that is derived by ϕ_j from Q, where

L_ℓ is not positioned further to the left than L_k, and

R_i is not positioned further to the right than R_j.

Then the distance between L_k and R_j is not larger than $m + 2n$, where $m = \underline{\max}\ \{\ \underline{\text{maxr}}(\phi_1), \underline{\text{maxr}}(\phi_2)\ \}$.

Proof of (i).

The situation is the best illustrated by the following picture:

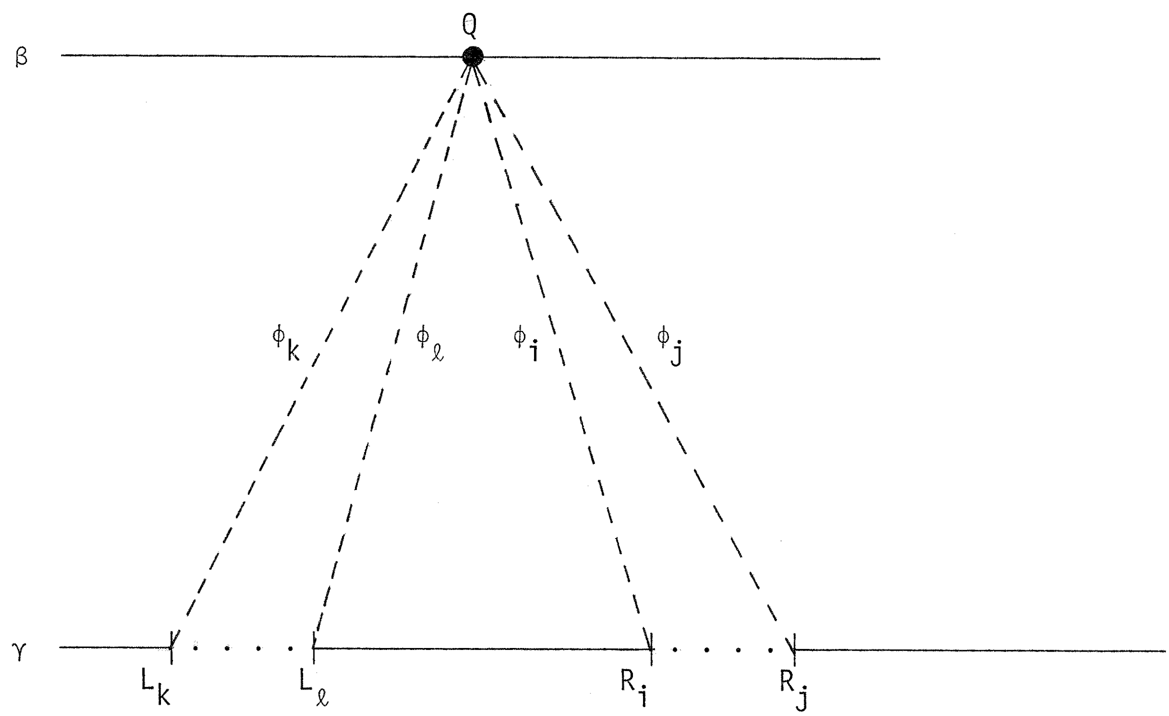


Fig. 1

Thus we have a "triangle" rooted at Q and with the base spreading from L_k to R_j. Since $X \in B(\phi_1,\phi_2)$ the distance between L_k and L_ℓ and the distance between R_i and R_j are both bounded by $b(X,\phi_1,\phi_2)$. On the other hand the distance between L_ℓ and R_i is bounded by m. Consequently the distance between L_k and R_j is bounded by m + 2n.

(ii) Let $\alpha \in Z$, $\omega = f_1(\alpha) = f_2(\alpha)$ and $\pi = g_1(\alpha) = g_2(\alpha) = h_1 f_1(\alpha) = h_2 f_2(\alpha)$. The situation is the best illustrated by the following picture:

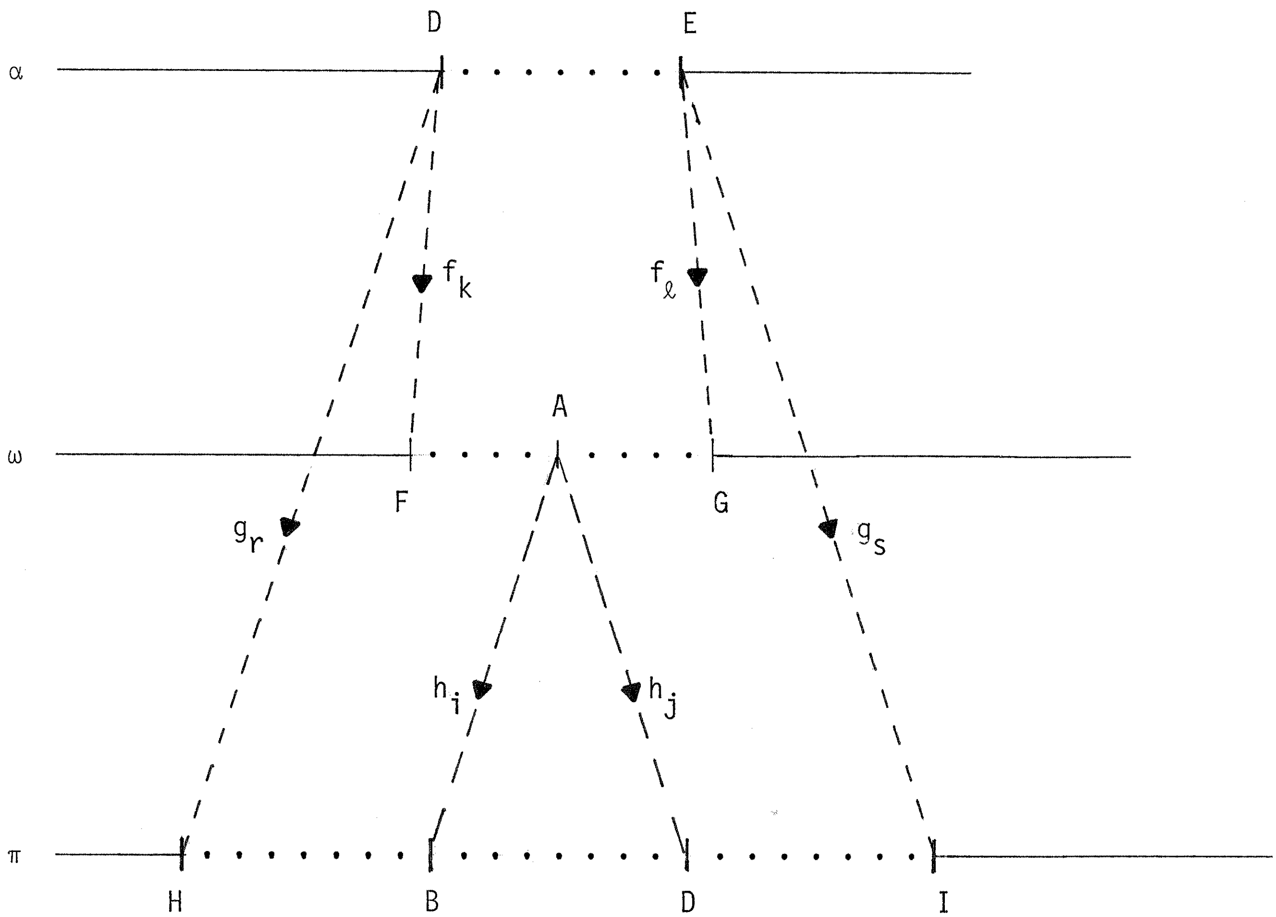


Fig. 2

where

$k,\ell,r,s,i,j \in \{1,2\}$,

A is an arbitrary occurrence in ω,

D,E are ancestors of A in α with respect to f_1 and f_2 where E is not to the left of D,

F is the leftmost among all the occurrences in ω that are derived from D by either f_1 or f_2,

G is the rightmost among all the occurrences in ω that are derived from E by either f_1 or f_2,

B is the leftmost among all the occurrences in π that are derived from A by either h_1 or h_2,

C is the rightmost among all the occurrences in π that is derived from A by either h_1 or h_2,

H is the leftmost among all the occurrences in π that is derived from D by either g_1 or h_2,

I is the rightmost among all the occurrences in π that is derived from E by either g_1 or g_2.

(iii) The distance between D and E is not larger than $u = 2\cdot(2\cdot n_f + m_f)$, where $n_f = b(Z,f_1,f_2)$ and $m_f = \max \{ \underline{maxr}(f_1),\underline{maxr}(f_2) \}$.

<u>Proof of (ii)</u>.

Since f_1 and f_2 are propagating it suffices to show that the distance between F and G is bounded by $2\cdot(2\cdot n_f + m_f)$. But this follows from (1), because the distance between F and G is formed by merging two "triangles" as in Fig. 1 with tops at D and E respectively whose bases overlap (at least at point A).

(iv) To prove the theorem it suffices to show that the distance between B and C is not larger than a certain constant dependent on f_1,f_2,g_1,g_2,h_1 and h_2 only.

However the distance between B and C is not larger than the distance between H and I. But (i) implies that the distance between H and I is not larger than $x\cdot(2\cdot n_g + m_g)$ where $n_g = b(Z,g_1,g_2)$, $m_f = \max\{ \underline{maxr}(g_1),\underline{maxr}(g_2) \}$ and x is the distance between D and E. Then from (iii) it follows that the distance between D and E is not larger than $v = u\cdot(2\cdot n_g + m_g)$ and consequently the distance between B and C is not larger than v. Since A was chosen to be an arbitrary occurrence in ω it means that h_1,h_2 are balanced on $f_1(Z) = f_2(Z)$ and as a matter of fact $b(f_1(Z),h_1,h_2)) \le 2\cdot(2\cdot n_f + m_f)\cdot(2\cdot n_g + m_g)$.

III. REGULAR TRUE ENVELOPES FOR EQUIVALENT DOL SYSTEMS

In this section we prove the main result of this paper: every two equivalent DOL systems have a regular true envelope. We start by examining the situation for the case of elementary DOL systems.

The following result was proved in [3].

Theorem 4. If h,g are elementary homomorphisms then $MID(h_1,h_2)$ is a regular language.

As a direct corollary we get the following result.

Corollary 2. Every two equivalent elementary DOL systems have a regular true envelope.

However in general MID(h,g) is not a regular language as is shown by the following example.

Example 1. Let $\Sigma = \{ a,b \}$ and let $h,g \in HOM(\Sigma,\Sigma)$ be defined by $h(a) = a, h(b) = aa, g(a) = aa, g(b) = a$. Then obviously $MID(h,g) = \{ \alpha \in \Sigma^* : \#_a(\alpha) = \#_b(\alpha) \}$ which is not a regular language.

Consequently to prove that every two equivalent DOL systems have a regular true envelope we will simplify not elementary DOL systems and reduce the problem to elementary DOL systems. To do it we need the following result which, for the restricted case of simplifiable homomorphisms, generalizes Lemma 8 from [3].

Theorem 5. Let $h_1,h_2 \in HOM(\Sigma,\Sigma)$ where at least one of h_1,h_2 is simplifiable. There exists a sequence $i_1,\ldots,i_k$ of elements from $\{1,2\}$ and homomorphisms f,p_1,p_2 such that $h_1h_{i_1}\ldots h_{i_k} = p_1f, h_1h_{i_1}\ldots h_{i_k} = p_2f$ and homomorphisms $p_1,p_2,h_1p_1,h_2p_2,fp_1,fp_2$ are elementary. Moreover if h_1,h_2 are effectively given then $i_1,\ldots,i_k$, f,p_1,p_2 can be effectively constructed.

Proof.

1) Assume that, e.g., h_1 is not elementary. Then $h_1 = g_1\hat{f}$ for some $\hat{f} \in HOM(\Sigma,\Gamma), g_1 \in HOM(\Gamma,\Sigma)$ where Γ is an alphabet such that $\#\Gamma < \#\Sigma$.

2) Let Θ be an alphabet such that

(i) $(\exists i_1,\ldots,i_k)_{\{1,2\}^+}(\exists f)_{HOM(\Sigma,\Theta)}(\exists g)_{HOM(\Theta,\Sigma)}[\ h_{i_1}\ldots h_{i_k} = gf\ ]$

and

(ii) $(\forall j_1,\ldots,j_\ell)_{\{1,2\}^+}[\text{ if } ((\exists\bar{f})_{HOM(\Sigma,\Delta)}(\exists\bar{g})_{HOM(\Theta,\Sigma)}[\ h_{j_1}\ldots h_{j_\ell} = \bar{g}\bar{f}\ ])$

then $(\#\Delta > \#\Sigma)\ ]$.

Take f,g satisfying (i) and set $p_1 = h_1g, p_2 = h_2g$.

Let $\tau_1 = h_1h_{i_1}\ldots h_{i_k}$ and $\tau_2 = h_2h_{i_1}\ldots h_{i_k}$, where $i_1,\ldots,i_k$ satisfies (i).

Then $\tau_1 = h_1gf = p_1f$ and $\tau_2 = h_2gf = p_2f$.

From the assumptions about Θ it follows that both p_1 and p_2 are elementary (because $p_1,p_2 \in HOM(\Theta,\Sigma)$). Moreover, because Θ is "minimal" in the sense of (i) and (ii), for every element τ from Sem($\{h_1,h_2\}$), τp_1 and τp_2 must be elementary, and so our choice of $i_1,\ldots,i_k,p_1,p_2$ and f satisfies the first part of the statement of the lemma.

The second part of this result is proved as follows. Let us generate systematically all sequences $i_1,\ldots,i_k$ from $\{1,2\}^+$. For each of them let us find whether or not there exists p_1,p_2,f satisfying conditions of the lemma. If we succeed we are done; if not we move to the next sequence. The first part of this proof guarantees that we will eventually succeed.

Now we can prove the main result of this paper.

Theorem 6. Every two equivalent DOL systems have a regular true envelope.

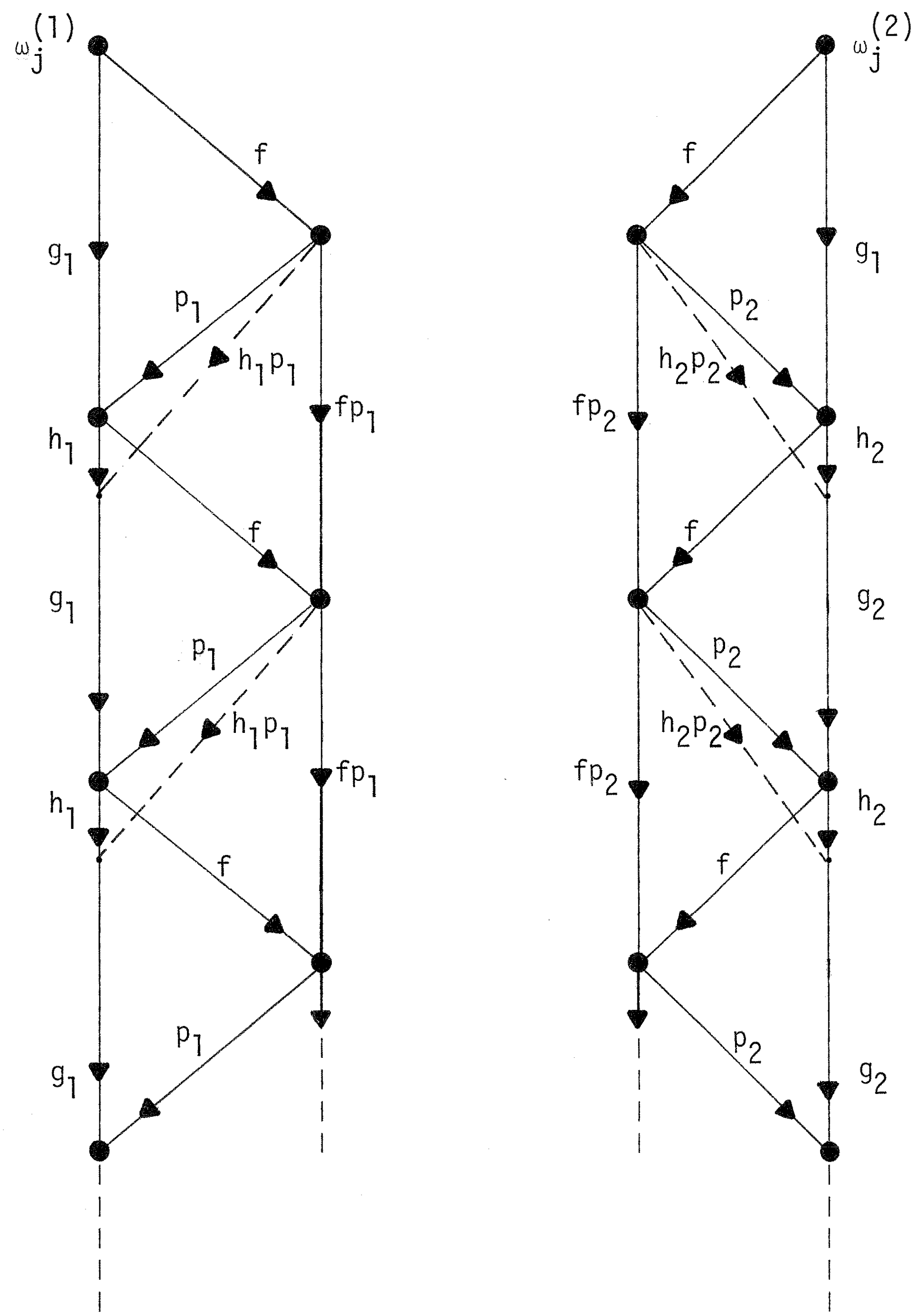
Proof.

Let $G_1 = (\Sigma,h_1,\omega), G_2 = (\Sigma,h_2,\omega)$ be two equivalent DOL systems with $K = L(G_1) = L(G_2)$, $E(G_1) = \omega_0^{(1)},\omega_1^{(1)},\ldots$ and $E(G_2) = \omega_0^{(2)},\omega_1^{(2)},\ldots$.

By Corollary 1 it suffices to demonstrate that $K \in BAL(h_1,h_2)$, because then $K \subseteq MID_{b(K,h_1,h_2)}(h_1,h_2)$.

(i) If G_1 and G_2 are elementary then the theorem is implied by Corollary 2.

(ii) Let at least one of G_1,G_2 be simplifiable. Let $i_1,\ldots,i_k,f,p_1,p_2$ satisfy the statement of Theorem 5. Let $g_1 = h_1h_{i_1}\ldots h_{i_k}$ and $g_2 = h_2h_{i_1}\ldots h_{i_k}$ and let for $1 \leq i \leq 2$, $0 \leq j \leq k+1$, $G_{i,j} = (\Sigma,g_i,\omega_j^{(i)})$. It is easy to see that $E(G_1) = E(G_2)$ if and only if, for every $0 \leq j \leq k+1$, $E(G_{1,j}) = E(G_{2,j})$. (A formal proof of this fact is provided in [3]). Thus is suffices to show that, for every $0 \leq j \leq k+1$, $M_j = L(G_{1,j}) = L(G_{2,j}) \in BAL(h_1,h_2)$. Theorem 5 implies that we have the following situation:



ω_j^(1)
ω_j^(2)
f
g1
p1
h1p1
fp1
h1
p2
h2p2
fp2
h2
g2

Since $L(G_{1,j}) = L(G_{2,j})$, the "inside" elementary DOL systems $H_{1,j} = (\Delta, fp_1, f(\omega_j^{(1)}))$ and $H_{2,j} = (\Delta, fp_2, f(\omega_j^{(2)}))$, are equivalent (where Δ is an alphabet through which g_1 and g_2 are simplified into f,p_1 and f,p_2 respectively).

Let $Z_j = L(H_{1,j}) = L(H_{2,j})$ and $\tau_j = E(H_{1,j}) = E(H_{2,j})$.

Since $p_1 =_{\tau_j} p_2$ and $h_1p_1 =_{\tau_j} h_2p_2$, by Corollary 2 we have that

$Z_j \in (BAL(p_1,p_2) \cap BAL(h_1p_1,h_2p_2))$. But then Theorem 3 implies that (note that p_1,p_2 are elementary and so Λ-free)

$p_1(Z_j) = p_2(Z_j) \in BAL(h_1,h_2)$. However M_j differs from $p_1(Z_j) = p_2(Z_j)$ by a finite set of words only and so $M_j \in BAL(h_1,h_2)$.

This completes the proof of the theorem.

IV. DISCUSSION

In this paper we have further investigated simplifications of homomorphisms studied already in [2] and [3]. We have further illustrated the usefulness of the elementary homomorphisms and some proof techniques concerning them by solving an open problem from [1]: we show that every two equivalent DOL systems have a true regular envelope. As indicated already in [1] this provides an alternative (to this in [1] and <u>also</u> to this in [2]) proof that the DOL sequence equivalence problem is decidable.

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FOOTNOTES

1) It is rather clear from [1] that some form of Theorem 2 and also Corollary 1 were known to the authors of [1]. However we are still convinced that the formal proof of Theorem 1 is needed here to keep this paper precise enough.

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