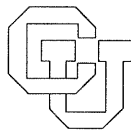Keyboards for Pure Music

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KEYBOARDS FOR PURE MUSIC

by

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0. <u>Summary</u>. We define a certain musical instrument M. The main idea is the organization and function of two basic keyboards described in Section 2. These keyboards permit one to play perfectly all conceivable consonant chords and a host of dissonant ones. The consonant chords form visible patterns, straight lines. The purpose of M is not to perform any existing music, but hopefully, to inspire a new kind of music.

1. <u>Preliminaries.</u> The rôle of consonance in music varies. Once consonant chords were regarded as the building blocks of music. Today music has a more subjective or cultural structure. The hearing of some patterns induces the ideas for others without acoustic motivation. I want to revisit the possibility of a more systematic use of acoustical consonance with the help of modern electronics. With such tools many new chords with beautiful sound can be made available to the composers and the performers.

A natural measure of consonance of an interval whose ratio of basic frequencies equals m/n, where m and n are integers larger than 1 and without common divisors, is the product mn; the smaller this product the better the consonance. Consonance depends of course also on the quality of the tones involved, but the above rule is corroborated by the following arithmetical observation. If p and q are simple tones with periods a_p and a_q (the periods are the inverses of the frequencies) and if a is the largest value such that $a_p = ma$ and $a_q = na$, where m and n are integers, then the period of the interval $p + q$ is mna (if no such a exists then $p + q$ is not periodic). There are 27 simple ratios m/n with $mn \leq 45$ (we disregard the difference between m/n and n/m as long as we talk of intervals, of course for triads and higher chords

this would not be possible). Those ordered according to the size of mn are the following 2/3, 2/5, 3/4, 2/7, 3/5, 2/9, 4/5, 3/7, 2/11, 3/8, 2/13, 4/7, 5/6, 3/10, 2/15, 3/11, 2/17, 5/7, 4/9, 2/19, 3/13, 5/8, 6/7, 3/14, 2/21, 4/11 and 5/9.

A tentative experiment made by the author seems to indicate that people with very good musical ears hear the consonance of all the above intervals when the qualities and intensities of the component tones are right. When the frequency of one of the tones is such that the ratio of frequencies approaches closely one of the above fractions one hears beats which slow down as the approximation of the fraction gets better. The smaller the product mn the more noticeable this pheonomenon seems to be. (For a general discussion of the hearing of consonance see [1] and references herein.)

The classic tempered scale which divides the octave into 12 equal intervals (half tones) yields good approximations to the fractions 2/3, 3/4, 2/9, 3/8, 2/17, 4/9 and 2/19 of the above list, they correspond to 7, 5, 26, 17, 37, 14 and 39 half tone intervals respectively. However the approximations are not perfect and one hears beats. Also, the ratios 2/5, 3/5, 4/5, 6/5, 2/15 and 5/8 are approximated by 16, 9, 4, 3, 35 and 8 half tones recpectively, but those approximations are bad and one has to get used to them to hear them as consonants. This shows limitations of equally tempered scales in representing intervals with simple ratios. (In fact among the divisions of the octave into k intervals with k < 19, k = 12 is the best from this point of view, and this division was independently discovered by the Chinese. See [2], [3] and [4] figure 39 and the accompanying text; figure 39 is poorly executed.)

We propose an instrument M which would permit to play perfectly all chords of the form $p_1 + \ldots + p_s$ where p_i are simple tones such that there exists a frequency ω such that the ratio of the frequency of each p_i to ω satisfies $mn \leq 45$. This may suggest a different kind of musical text. (If it proves interesting it could be reasonable to invent a new notation for this music.)

The controls of M will consist of three keyboards and a pedal: keyboard I controls the basic frequency, keyboard II plays, keyboard III controls the quality of tones and the pedal controls the loudness. We limit our description to the form and function of those controls. We hope that this will be enough for the imaginative constructor.

2. <u>Main keyboards</u>. The main keyboards I and II are both shown on Figure. In both the keys correspond to integer lattice points (m,n) on the plane with $0 < mn \leq 45$, and they are arranged in a hexagonal pattern. The keys are labeled by the simplified fractions m/n (not by the original pairs (m,n)). I differs from II in that all the keys with label 1 are deleted.

Keyboard I (the control keyboard) works as follows. M is a machine which can be in various states. The states of M are denoted by positive real numbers. If M is in state x and we press the key m/n of I then M goes into state mx/n. Releasing a key does not change the state of M, it stays in the state which it reached when the key was pressed. If we press simultaneously several keys of I the effect is the same as after pressing them one by one. Thus e.g. pressing 2/3 and 5/4 in I has the same effect as pressing 5/6. Pressing m/n and n/m simultaneously has no effect at all.

Keyboard II (the playing keyboard) operates as follows. If M is in state x and the key m/n of II is pressed then M plays a simple tone with basic fequency mx/n. If several keys of II are pressed then M plays the sum of the corresponding tones.

The keys of I and II can be pressed independently so that e.g. if a key of I is pressed at a time a key of II is being pressed the frequency the tone which is being played changes accordingly.

Notice that if we press some keys in one line parallel to the right slope R of keyboard II we get a major chord e.g. $\frac{2}{5}:\frac{3}{5}:1$. If we press some keys in one line parallel to the left slope L we get a minor chord e.g. $\frac{5}{2}:\frac{5}{3}:1$. Notice the five lines of keys $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, separating the octaves. Thus the configurations of keys playing consonant chords are much easier to locate than on a piano keyboard.

For any positive real number x the set of numbers $2^i x/3^j$ where i, j = 0,1,2,... is dense in the positive part of the real line. Hence M must be able to play simple tones of practically arbitrary basic frequencies (within a part of the audible range). Therefore M requires an electronic synthesizing unit to produce those tones, unlike the classical instruments which have a fixed discrete scale.

The keys should be located in holes in the frames of the keyboards evenly with the plane of the frames, so that the fingers of the performer do not depress accidentally keys adjacent to the intended ones.

3. <u>The quality keyboard</u>. The quality of the simple tones played by M (when we press the keys of II) will be controlled by the keyboard III and a pedal. Assuming that the ears care only about the first 10 harmonics (see [1], §4,8) we let III consist of 10 revolving knobs. The k-th knob and the pedal determine the amplitude α_k of the k-th harmonic. That is when a key of II requests a simple tone of frequency ω then M plays the following one

$$p(t) = \lambda \sum_{k=1}^{10} \alpha_k \sin(2\pi k\omega t),$$

where λ is controlled by the pedal, and α_k by the k-th knob.

It may take too much time and attention to set properly the 10 knobs each time we want to change the quality. The following modification of III will allow us to perform this in one movement of a finger. Imagine a vertical rectangle of 10×20 keys. When we move a finger across it along a horizontal straight or wavy line, crossing once each column of keys, the keys that are touched become depressed (or glow) and all the keys which were depressed before are released (or stop glowing). The k-th column of keys controls α_k, which is available in 20 different sizes; this should be enough to achieve any desired quality.

4. <u>General outline of M</u>. M should consist of 4 main units: (1) The keyboards I, II and III and a pedal controlling loudness. (2) An electronic calculator which instantly performs the multiplications necessary to establish the state x of M after each of the changes ordered by keyboard I and also computes the, say s, frequencies

$\omega_r = m_r x/n_r$ $(r = 1,\ldots,s)$ requested by keyboard II. (3) A unit which, given the sequence $\omega_1,\ldots,\omega_s$, and the sequence $\alpha_1,\ldots,\alpha_{10}$ ordered by keyboard III and a coefficient of loudness λ ordered by the pedal produces a current of appropriate intensity $f(t)$. (4) A loudspeaker transforming $f(t)$ into the tone
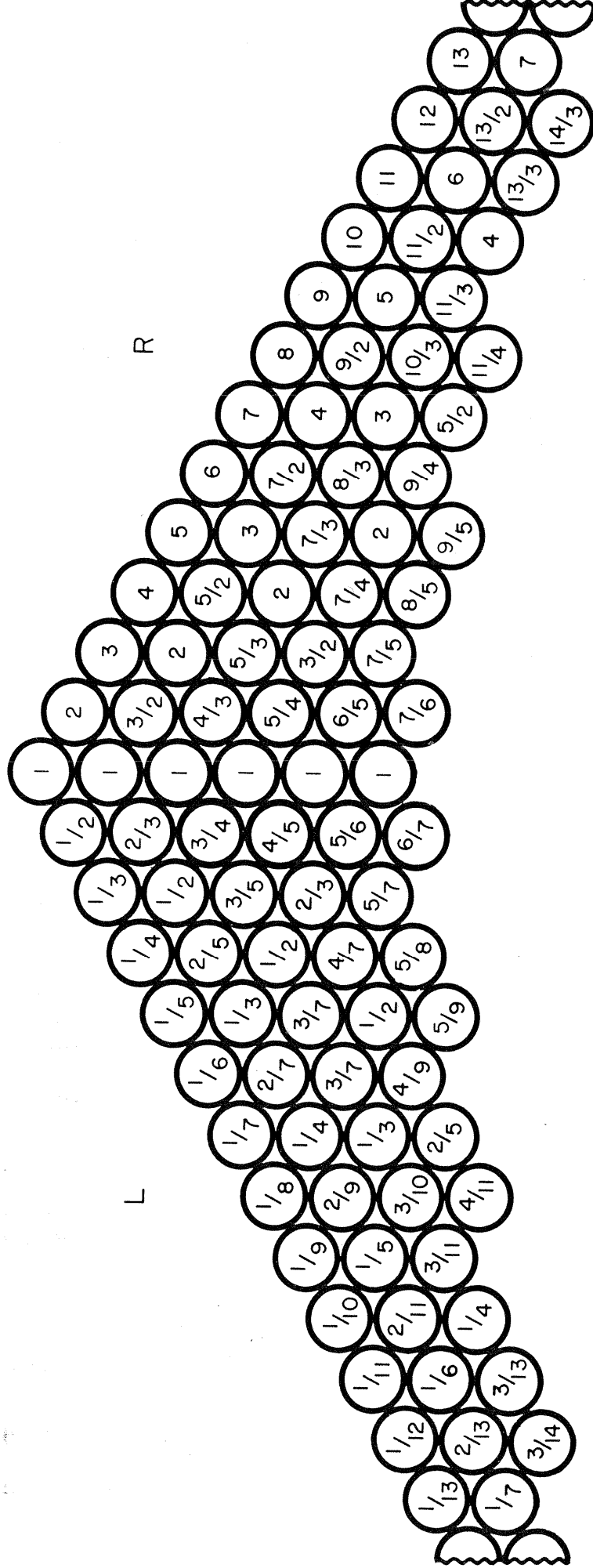
$$p(t) = \lambda \sum_{k=1}^{10} \alpha_k \sum_{r=1}^{s} \sin(2\pi k \omega_r t)$$

Notice that since the ratios of the numbers ω_r are rational, the terms $\sin(2\pi k \omega_r t)$ are either orthogonal or equal. Therefore no cancellations or partial cancellations occur among them. It may be interesting to shift each of them at random in phase or to substitute p by p/s to reduce somewhat or to eliminate completely the dependence of loudness on the number s of the keys of II which are pressed.

5. <u>An instrument for producing rhythms</u>. Another possible application of keyboards I and II is the following. Pressing a key n/m of II makes a sequence of equally spaced drum beats. The numbers of beats per second is nx/m, where, as above, x is the state, except that now much smaller x are to be used. Also, as above, if several keys of II are pressed the corresponding sounds are superimposed. The resulting sound is called a rhythm. It is clear that such an instrument would permit to play interesting sequences of rhythms. This could be worthwhile even without keyboard I, i.e., without control over the state x. The rhythm could be more interesting if, the keys of II have appropriate phase shifts relative to each other, so that no simultaneous beats occur. One can also think of a different arrangement in which the keys of II order not only a different time spacing but also a different pitch.

REFERENCES

[1]   Juan G. Roederer, Introduction to the physics and psychophysics
       of music, Springer-Verlag, New York 1973.

[2]   A. L. Leigh Silver, Musimatics or the nun's fiddle, The Amer.
       Math. Monthly 78 (1971), 351-357.

[3]   Jan Mikusiński, Matematyka dźwieku i skali muzycznej, Problemy
       10 (1954), 668-675. (in Polish)

[4]   H. Steinhaus, Mathematical Snapshots, Oxford Univ. Press, New
       York, Oxford Univ. Press 1960.

FIGURE