ON THE STRUCTURE OF POLYNOMIALLY

BOUNDED DOL SYSTEMS

by

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#CU-CS-095-76   September, 1976

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working report

ABSTRACT

A notion of a DOL system with rank is introduced. It provides
a structural characterization of polynomially bounded DOL systems.
Several consequences of such a characterization are studied.

INTRODUCTION

L systems (see, e.g., [4], [6] or [8]) have recently gained considerable attention in both formal language theory and theoretical biology. One of the most interesting and physically best motivated topics in the theory of L systems is that of "local versus global properties". It is concerned with explaining on local level (sets of productions) global properties (i.e., those properties of the language or the sequence generated by an L system whose formulation is independent on the L system itself).

This paper is concerned with the topic of "local versus global properties" in the case of DOL systems. It provides a structural characterization of those DOL systems whose growth functions are polynomially bounded. In this sense it continues the work from [9] and [11].

We present here a full and extended version of an extended abstract [2]. We also prove some results not stated in [2]. On the other hand we do not prove here Corollary 1 from [2], because in the meantime the equivalence problem for arbitrary DOL systems was proved decidable (see [1]).

PRELIMINARIES

We assume the reader to be familiar with basics of formal language theory (see, e.g., [10]). We use mostly the standard language-theoretic notation. The following is perhaps worth mentioning.

If A is a set then $P(A)$ denotes the set of subsets of A. We use N, N^+, R, R^+ to denote the sets of nonnegative integers, positive integers, real numbers and positive real numbers respectively. In this paper we will consider polynomials over N with integer coefficients and a positive coefficient at the highest power. We will use $Pol(k)$ to denote the set of such polynomials of degree k. For a function f from a set X into itself we use f^i to denote the i-folded composition of f with itself.

For a language K, *Length* K denotes the length set of K and for a positive integer q, $less_q K = \{n \in Length\ K : n < q\}$.

For a word x, $|x|$ denotes its length and *min* x denotes the set of letters occurring in x. If V is a set of letters then $\#_V x$ denotes the number of occurrences of elements from V in x and $\phi_V(x)$ denotes the word resulting from x by erasing all letters in x that are not in V.

Now we recall some basic notions concerning DOL systems.

Definition 1. A DOL *system* is a triple $G = <\Sigma, \delta, \omega>$ where Σ is a finite alphabet (the *alphabet of* G), ω is a nonempty word over Σ (the *axiom of* G) and δ is a homomorphism from Σ into Σ^* (called the *transition function of* G).

Definition 2. Let $G = <\Sigma, \delta, \omega>$ be a DOL system.
1) The *sequence of* G, denoted as $E(G)$, is the sequence $\omega_0, \omega_1, \ldots$ of words over Σ such that $\omega_0 = \omega$ and $\omega_i = \delta^i(\omega)$ for every $i \geq 0$.

2) The *growth function of* G, denoted as f_G, is a function from the non-negative integers into itself such that $f_G(n)=|\omega_n|$ for every $n \geq 0$. For a word z in Σ^+, $f_{G,z}$ denotes the growth function of the DOL system $G_z=<\Sigma,\delta,z>$ and if $V \subseteq \Sigma$ then $f_{G,z,V}$ is a function from nonnegative integers into itself such that $f_{G,z,V}(n)=\#_V \delta^n(z)$. We say that G is *polynomially bounded* if there exists a polynomial p such that, for every $n \geq 0$, $f_G(n) \leq p(n)$.

3) The *language of* G, denoted as L(G), is defined by $L(G)=\{x \varepsilon \Sigma^*: \delta^i(\omega)=x$ for some $i \geq 0\}$.

4) Let $a \varepsilon \Sigma$. We say that a is *useful* if $a \varepsilon min \, \delta^i(\omega)$ for some $i \geq 0$. We say that a is *expansive* if $\delta^i(a)=\alpha_1 a \alpha_2 a \alpha_3$ for some $i \geq 0$ and $\alpha_1,\alpha_2,\alpha_3$ in Σ^*. (We cal G *expansive* if it contains a useful expansive letter.) We say that a is *recursive* if $\delta^i(a)=\alpha_1 a \alpha_2$ for some $i \geq 0$ and α_1,α_2 in Σ^*.

If not clear otherwise, we consider only DOL systems that generate infinite languages, because otherwise the problems we consider become trivial.

Definition 3. Let $G=<\Sigma,\delta,\omega>$ be a DOL system and let K be a language. The *spectrum of* G *with respect to* K, denoted as *Spec(G,K)*, is defined by $Spec(G,K)=\{n \varepsilon N: \delta^n(\omega) \varepsilon K\}$.

The following result is a special case of a more general theorem from [3].

Lemma 1. If G is a DOL system and K a regular language, then *Spec(G,K)* is an ultimately periodic set.

<u>Defintion 4</u>. Let $G=<\Sigma,\delta,\omega>$ be a DOL system and m a positive integer. The m-*decomposition* of G, denoted as $Dec_m G$, is a finite set $G_0,\ldots,G_{m-1}$ of DOL systems such that, for $0\le i\le m-1$, $G_i=<\Sigma,\overline{\delta},\omega_i>$ where, for every a in Σ, $\overline{\delta}(a)=\alpha$ if and only if $\delta^m(a)=\alpha$. The set U of DOL systems is called a *decomposition of* G if there exists an m, such that $U=Dec_m G$.

<u>Definition 5</u>. Let $G=<\Sigma,\delta,\omega>$ be a DOL system. G is *well-sliced* if

1) $(\forall a)_\Sigma (\forall m,n)_{N^+} [min\ \delta^m(a)=min\ \delta^{(n)}(a)]$,

2) $(\forall a)_\Sigma (\forall V)_{P(\Sigma)} [if\ f_{G,a,V}$ is bounded *then*
$$(\forall m,n)_{N^+} [\phi_V(\delta^m(a))=\phi_V(\delta^n(a))]]$$

<u>Definition 6</u>. A decomposition of a DOL system is *well-sliced* if each of its elements is well-sliced.

<u>Lemma 2</u>. Let $G=<\Sigma,\delta,\omega>$ be a DOL system. There exists a well-sliced decomposition of G.

<u>Proof</u>

First of all it is well known, see [7] or [4], that the sequence $min\ \omega,\ min\ \delta(\omega),\ min\ \delta^2(\omega),\ldots$ is ultimately periodic. (Notice that this is also an immediate corollary from Lemma 1). Let t be the threshold of it and p its period. Let m_0 be a positive integer larger than t and divisible by p.

Now let a be in Σ and let V be a subset of Σ. Let us assume that $f_{G,a,V}$ is bounded, in other words that the sequence $\phi_V(\delta(a)),\phi_V(\delta^2(a)),\ldots$ is built up from a finite number of words only. Let the set of these words be $b_{11}\ldots b_{1n_1}, b_{21}\ldots b_{2n_2},\ldots,b_{k1}\ldots b_{kn_k}$ with $b_{ij}\epsilon V$ for $1\le i\le k,\ 1\le j\le n_i$. But for $1\le i\le k$,

$$Z_i = (\Sigma \backslash V)^* \{b_{i1}\}(\Sigma \backslash V)^* \{b_{i2}\} \ldots (\Sigma \backslash V)^* \{b_{in_i}\}(\Sigma \backslash V)^*$$

is a regular set and so, by Lemma 1, $Spec(G_a, Z_i)$ is an ultimately periodic set. Let $m_{a,i}$ be a positive integer larger than the threshold of $Spec(G_a, Z_i)$ and divisible by the period of $Spec(G_a, Z_i)$.

Finally let m be the least common multiple of all the numbers $m_{a,2}$ for all a in Σ and the number m_0. Then, obviously, the m-decomposition of G is well-sliced.

<u>Definition 7.</u> Let $G=<\Sigma, \delta, \omega>$ be a DOL system and let a be in Σ. Then we say that

1) a is of *growth type* 0 (*in* G) if $\Lambda \varepsilon L(G_a)$, and

2) a is of *growth type* 1 (*in* G) if $\Lambda \not\in L(G_a)$ and $L(G_a)$ is finite.

Now comes the definition of the basic notion of this paper.

<u>Definition 8.</u> Let $G=<\Sigma, \delta, \omega>$ be a DOL system.

(1) The *rank of a letter* a *in* G, denoted as $rank_G(a)$ is defined inductively as follows.

   (i) If a is of growth type 0 or 1 in G then $rank_G(a)=0$,

   (ii) For $n \geq 1$, let δ_n denote the restriction of δ to $\Sigma_n = \Sigma \backslash \{b : rank_G(b) \leq n\}$ and let $G_n = <\Sigma, \delta_n, \omega>$.
      If a is of growth type 1 in G_n, then $rank_G(a)=n$.

(2) If z is a word consisting of letters with a rank only, then the *rank of* z *in* G, denoted as $rank_G(z)$, is defined as the highest rank of a letter occurring in G.

(3) We say that G is a DOL *system with rank* if every useful letter in G has a rank. In that case the *rank of* G, denoted as $rankG$, is defined as the highest of the ranks of useful letters in G.

The following two results follow directly from the definitions.

Lemma 3. It is decidable whether an arbitrary DOL system has a rank.

Lemma 4. Let $G=<\Sigma,\delta,\omega>$ be a DOL system with rank. Let K be a decomposition of G and let $H=<V,\bar{\delta},\rho>$ be in K. Then for every b in V, $rank_H(b)=k$ *if and only if* $rank_G(b)=k$.

RESULTS

In this section we investigate properties of DOL systems with rank.

Lemma 5. Let $G=<\Sigma,\delta,\omega>$ be a well-sliced DOL system with rank. Then
$(\forall z)_{\Sigma^+}[\textit{if } rank_G(z)=k \textit{ then}$

$\quad\quad (\exists p_z)_{Pol(k)} (\forall n)_{N^+}[\textit{if } n>2k \textit{ then } f_{G,z}(n)=p_z(n)].$

Proof.

The proof goes by induction on $rank_G(z)$.

(1) Let $rank_G(z)=0$.

(i.1) If $z=a\epsilon\Sigma$, then the result follows immediately from the second condition on a well-sliced DOL system. (see Definitions 5 and 6, take $V=\Sigma$).

(i.2) If $z=a_1...a_t$ with $a_1,...,a_t\epsilon\Sigma$, then, for every $n>0$,

$$f_{G,z}(n)=\sum_{i=1}^{t} f_{G,a_i}(n) \text{ and so by (i.1) the result holds.}$$

(ii) Let us assume that the result holds for every z in Σ^+ with $rank_G(z)\leq k-1$.

(iii) Let $rank_G(z)=k$.

(iii.1) Let $z=a\epsilon\Sigma$.

(iii.1.1) Let a be a recursive letter.

Then clearly $\delta(a)=\alpha a\beta$ with $rank_G(\alpha\beta)<k$ and $\alpha\beta\neq\Lambda$. Hence, for every positive integer n,

$\delta^n(a)=\delta^{n-1}(\alpha)...\delta(\alpha)\alpha a\beta\delta(\beta)...\delta^{n-1}(\beta),$

and so if $n>2k-1$, then

$$f_{G,a}(n) = |a| + |\alpha\beta| + f_{G,\alpha\beta}(1) + \ldots + f_{G,\alpha\beta}(2k-2) + \sum_{i > 2k-2}^{n-1} f_{G,\alpha\beta}(i)$$

If we set now C to be the positive integer constant equal to $1 + |\alpha\beta| + f_{G,\alpha\beta}(1) + \ldots + f_{G,\alpha\beta}(2k-2)$, then we get

$$f_{G,a}(n) = C + \sum_{i > 2k-2}^{n-1} f_{G,\alpha\beta}(i) \ .$$

Since $rank_G(\alpha\beta) < k$, the above summation yields a polynomial of degree k and so the result holds.

(iii.1.2) Let a be a nonrecursive letter.

Then, clearly, $\delta(a) = \alpha_0 b_1 \alpha_1 \ldots b_s \alpha_s$ with $b_1, \ldots, b_s \epsilon \Sigma \setminus \{a\}, \alpha_0, \ldots, \alpha_s \epsilon \Sigma^*$, $rank_G(b_1 \ldots b_s) = k$ and $rank_G(\alpha_0 \ldots \alpha_s) < k$. From the second condition on a well-sliced DOL system (see Definitions 5 and 6, take V equal to the set of all letters in Σ of rank k) it follows that each $b_i, 1 \leq i \leq s$, is a recursive letter.

For every positive integer n,

$$f_{G,a}(n) = f_{G,\alpha_0 \ldots \alpha_s}(n-1) + f_{G,b_1}(n-1) + \ldots + f_{G,b_s}(n-1),$$

where by the inductive assumption $f_{G,\alpha_0 \ldots \alpha_s}$ is a polynomial of degree smaller than k, and by (iii,1.1) each $f_{G,b_j}, 1 \leq j \leq s$, is a polynomial of degree k.

Consequently if we take $n > 2s$, hence $n-1 > 2s-1$, then the result follows from (iii.1.1).

(iii.2) If $z \epsilon \Sigma^+$, then we complete the argument in the same way as in (i.2).

Remark. Please note that the bound n>2k results from the following. For the letters of rank 0 we get a polynomial description of their growth from the first step on. If a letter b is recursive of rank 1, hence $\delta(b)=\alpha b\beta$, then we get a polynomial description of its growth after rewriting it at least once *and* after we get a polynomial description of growth of $\alpha\beta$ (which is of rank 0). If a letter b is nonrecursive of rank 1, then it takes one step for it to introduce recursive letters and one step for each recursive letter to introduce itself with some letters of rank 0 for which it takes again one step to settle for a polynomial description of its growth. Iterating this argument we got that, indeed, it takes more than 2k steps for a letter of rank k before its polynomial description settles down.

As a direct corollary from Lemma 5 we get the following result.

Theorem 1. If G is a well-sliced DOL system of rank k then

$$(\exists p_G)_{Po\ell(k)}(\forall n)_{N^+}[i6\ n>2k\ \text{then}\ f_G(n)=p_G(n)].$$

Theorem 2. Let G be a DOL system of rank k. Then

$$(\exists m)_{N^+}(\exists p_0,\ldots,p_{m-1})_{Po\ell(k)}(\forall n)_N$$

$$[i6\ n=t\cdot m+u\ \text{with}\ 0\leq u<m\ \text{and}\ t>2k\ \text{then}\ f_G(n)=p_u(t)].$$

Proof.

It follows directly from Lemma 2, Lemma 4 and Lemma 5.

Corollary 1. Let G be a DOL system of rank k. Then

$$(\exists c_1)_{R^+}(\exists c_2)_{N^+}(\forall n)_{N^+}[c_1\cdot n^k\leq f_G(n-1)\leq c_2\cdot n^k].$$

Proof.

Let $G=<\Sigma,\delta,\omega>$. By Theorem 2 there exists $m\geq 1$ and m polynomials

$p_0, \ldots, p_{m-1}$ of degree k such that if t>2k and n=t·m+u with $0 \leq u < m$ then $f_G(n) = p_u(t)$. Hence if n>2km+u, then $f_G(n) = p_u(\frac{n-u}{m})$. But $p_u(\frac{n-u}{m})$ is a polynomial on n of degree k; let us denote this polynomial by $g_u(n)$.

Let MAX(n) be the polynomial of degree k resulting by taking as the coefficient for $n^{\ell}, 0 \leq \ell \leq k$, the maximal of all coefficients of n^{ℓ} among all $g_u(n)$, $0 \leq u < m$. Similarly let MIN(n) be the polynomial of degree k resulting by taking as the coefficient for n^{ℓ}, $0 \leq \ell \leq k$, the minimal of all coefficient of n^{ℓ} among all $g_u(n)$, $0 \leq u < m$.

1) Let $Z(n) = MAX(n) + \max\{f_G(i) : 0 \leq i \leq 2km + (m-1)\}$.

Obviously $f_G(n) \leq Z(n)$ for all $n \geq 0$.

We have $Z(n) = a_0 n^k + Z_1(n)$, where $Z_1(n)$ is a polynomial of degree smaller than k. Let n_0 be the smallest nonnegative interger such that for every $n \geq n_0$, $a_0 n^k \geq Z_1(n-1)$.

Now let us choose

$c_2 = (2a_0 + 1)(abs(Z(1)) + 1) \cdot (abs(Z(2)) + 2) \cdot \ldots \cdot (abs(Z(n_0)) + 1)$,

and let us check that indeed, for every $n \geq 1$

$f_G(n-1) \leq c_2 \cdot n^k$.

(i) If $n \geq n_0$ then $a_0 n^k \geq Z_1(n)$ and since $(abs(Z(1)) + 1) \cdot \ldots \cdot (abs(Z(n_0)) + 1)$

is not smaller than 1, then indeed

$f_G(n-1) \leq Z(n-1) \leq 2a_0 n^k \leq c_2 \cdot n^k$.

(ii) If $n < n_0$ then $a_0 n^k < Z_1(n-1)$. Then, indeed

$f_G(n-1) \leq 2 \cdot Z_1(n-1) \leq c_2$.

2) Let T(n)=MIN(n).

Obviously $f_G(n) \geq T(n)$ for all $n \geq 0$.

We have $T(n) = a_0 n^k + T_1(n)$, where $T_1(n)$ is a polynomial of degree smaller than k. Let n_0 be the smallest nonnegative integer such that $n_0 > 2km + (m-1)$ and for every $n \geq n_0$, $\frac{a_0}{2} n^k < T(n-1)$.

Now let us choose

$$c_1 = \frac{a_0}{2 \cdot (a_0+1) \cdot n_0^k}$$

and let us check that indeed, for every $n \geq 1$,

$$f_G(n-1) \geq c_1 \cdot n^k.$$

(i) If $n \geq n_0$, then $f_G(n-1) \geq T(n-1) > \frac{a_0}{2} \cdot n^k$, and so indeed

$$f_G(n-1) \geq c_1 \cdot n^k.$$

(ii) If $n < n_0$, then

$$c_1 \cdot n^k < 1 \leq f_G(n-1).$$

Remark. Please note that since $E(G)$ is counted starting with 0 (it is $E(G) = \omega_0, \omega_1, \ldots$), the statement of the above result is in the form $c_1 \cdot n^k \leq f_G(n-1) \leq c_2 \cdot n^k$ rather than in the form $c_1 \cdot n^k \leq f_G(n) \leq c_2 \cdot n^k$.

Now we get a characterization of polynomially bounded DOL systems.

Theorem 3. A DOL system G is of rank k if and only if k is the minimal degree of a polynomial p, such that, for every n, $p(n) \geq f_G(n)$.

Proof.

It follows from Corollary 1 that if suffixes to prove that if a DOL system is polynomially bounded then it has a rank.

This is proved as follows.

Let us assume that $G = \langle \Sigma, \delta, \omega \rangle$ is a polynomially bounded system but it does not have a rank. Then constructing, as in Definition 8, consecutive systems $G_0, G_1, \ldots$ we must arrive at the system $G_j = \langle \Sigma, \delta_j, \omega \rangle$ such that no letter in G_j is of growth type 0 or 1. Following [12] let us consider the equivalence relation $\sim$ on Σ defined by $a \sim b$ if and only if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ in Σ^* such that $a \overset{*}{\Longrightarrow} \alpha_1 b \alpha_2$ and $b \overset{*}{\Longrightarrow} \beta_1 a \beta_2$. Let us consider now an equivalence class A of $\sim$ which is a minimal one (it is no element of this class can introduce an element of another

class). Let $x \in A$. Since x derives in G_j an infinite language then there exist a positive integer s, words z_1, z_2, z_3 and a letter y in A such that $\delta^s(x) = z_1 y z_2 y z_3$. Since y is in A, there exist a nonnegative integer r and words $\bar{z}_1, \bar{z}_2$ such that $\delta^r(y) = \bar{z}_1 x \bar{z}_2$. Consequently there exist words u_1, u_2, u_3 such that $\delta^{r+s}(x) = u_1 x u_2 x u_3$. Thus x is an expansive letter and consequently, by [9], G is not polynomially bounded.

Let us turn now to the length sets of images of DOL languages through nonerasing homomorphisms.

Theorem 4. Let G be a DOL system of rank k, k>0, and let ψ be a Λ-free homomorphism. Then

$$(\exists d_1, d_2)_{R^+} (\forall q)_{N^+} [d_1 \cdot q^{\frac{1}{k}} \leq less_q \psi(L(G)) \leq d_2 \cdot q^{\frac{1}{k}}].$$

Proof.

1) We will first prove that

$$(\exists d_1, d_2)_{R^+} (\forall q)_{N^+} [d_1 \cdot q^{\frac{1}{k}} \leq less_q L(G) \leq d_2 \cdot q^{\frac{1}{k}}]$$

This is done as follows.

Let $G = \langle \Sigma, \delta, \omega \rangle$. It was proved in [5] that, for every natural number n, L(G) does not contain more that m! elements of the length n, where $m = \#\Sigma$.

Let $d_1 = \frac{1}{c_2 \cdot m!}$ and $d_2 = \frac{1}{c_1}$ where c_1 and c_2 are constants from the statement of Corollary 1. Also to avoid strange indexing let us consider that the elements of $E(G)$ are numbered starting with 1 (that is $E(G) = \omega_1, \omega_2, \ldots$). Then from Corollary 1 we have that

$$(\forall n)_{N^+} [c_1 \cdot n^k \leq f_G(n) \leq c_2 \cdot n^k].$$

Since $c_1 \cdot n^k \leq f_G(n)$, *if* $n \geq \dfrac{q^{\frac{1}{k}}}{c_1}$ *then* $f_G(n) \geq q$.

Consequently *if* $f_G(n) < q$ *then* $n < \dfrac{q^{\frac{1}{k}}}{c_1}$ and so

$$\mathcal{less}_q L(G) < \frac{1}{c_1} \cdot q^{\frac{1}{k}} \ .$$

Since $f_G(n) \leq c_2 \cdot n^k$, *if* $n \leq \dfrac{1}{c_2} \cdot q^{\frac{1}{k}}$ *then* $f_G(n) \leq q$, and so

$$\mathcal{less}_q L(G) \geq \frac{1}{-c_2 \cdot m!} \cdot q^{\frac{1}{k}}$$

Hence the result holds.

2) Now we will prove the theorem.

Let $\psi : \Sigma \longrightarrow \Theta^+$. Let $\overline{G} = \langle V, \overline{\delta}, \omega \rangle$ be a DOL system such that

$V = \Sigma \cup Q \cup \Sigma \times \Theta$,

for a in Σ, *if* $\delta(a) = \Lambda$, *then* $\overline{\delta}(a) = \Lambda$,

if $\delta(a) = b_1 \ldots b_k, b_1, \ldots, b_k \varepsilon \Sigma$, *then* $\overline{\delta}(a) = (a, b_1) b_2 \ldots b_k$,

for a in Θ, $\overline{\delta}(a) = \Lambda$,

for $(a,c) \varepsilon \Sigma \times \Theta$, $\overline{\delta}((a,c)) = \delta(a)$.

Let us consider the 2-decomposition of $\overline{G}$, $\mathcal{Dec}_2 \overline{G} = \{G_1, G_2\}$ where

the axiom of G_1 is ω and the axiom of G_2 is $\overline{\delta}(\omega)$. Then

(i) The derivation tree of $E(G)$ is identical to the derivation tree of

   $E(G_1)$, hence $\mathcal{rank}G = \mathcal{rank}G_1$ and consequently, by Lemma 4

   $\mathcal{rank}G = \mathcal{rank}G_1 = \mathcal{rank}G_2 = \mathcal{rank}\overline{G}$.

(ii) The length sequence of $E(G_2)$ is identical to $|\phi(\omega)|, |\phi(\delta(\omega))|, \ldots$

   and so, for every positive integer q

   $\mathcal{Less}_q(\phi(L(G))) = \mathcal{less}_q(L(G_2))$.

Now the theorem follows from (i), (ii) and (1).

Let us contrast now the situation with DOL systems without rank.

Theorem 5. Let G be a DOL system without rank and let ψ be a

Λ-free homomorphism. Then

$$(\exists \alpha,\beta)_{R^+}(\forall q)_{N^+}[\alpha \cdot \log_2 q \leq \ell e \delta \delta_q \psi(L(G)) \leq \beta \cdot \log_2 q]$$

Proof.

1) First we will prove that

$$(\exists \alpha,\beta)_{R^+}(\forall q)_{N^+}[\alpha \cdot \log_2 q \leq \ell e \delta \delta_q L(G) \leq \beta \cdot \log_2 q]$$

This is done as follows.

Let $G=<\Sigma,\delta,\omega>$.

(i) $(\exists c,d)_{R^+}(\forall n)_{N^+}[2^{d \cdot n} \leq f_G(n) \leq 2^{c \cdot n}]$.

This is proved as follows.

(i.1) Let t be not smaller than the maximal length of $\delta(a)$, $a\varepsilon\Sigma$,

nor the length of the axiom ω. Then clearly $f_G(n) \leq t^n = 2^{n \cdot \log_2 t}$ and

so if we set $c=\log_2 t$ then indeed $f_G(n) \leq 2^{c \cdot n}$.

(i.2) Since G is a system without rank, it follows from Definition 8

(in the same way as in the proof of Theorem 3) that G contains a useful

expansive letter. Applying Lemma 1 and the technique demonstrated in

the proof of Lemma 2 we can find an m-decomposition of G such that

each element $H=<\Sigma,\overline{\delta},\overline{\omega}>$ of $\mathcal{Dec}_m G$ will be such that for each expansive

letter a in it, $\delta(a)=\alpha a \beta a \gamma$ for some α,β,γ in Σ^*. Then clearly

$f_H(n) \geq 2^n$ which implies that $f_G(n) \geq 2^{\frac{1}{m} \cdot n}$.

Hence if we set $d=\frac{1}{m}$, then indeed $2^{d \cdot n} \leq f_G(n)$.

(ii) From (i) it follows that if $n \leq \frac{1}{c} \cdot \log_2 q$ then

$$f_G(n) \leq 2^{c \cdot n} \leq 2^{c \cdot \frac{1}{c} \cdot \log_2 q} = 2^{\log_2 q} = q$$

It was proved in [5] that, for every natural number n, L(G) does not contain more than m! elements of the length n, where m=#Σ. Thus $\mathit{less}_q L(G) \geq \frac{1}{c \cdot m!} \cdot \log_2 q$ and so if we take $\alpha = \frac{1}{c \cdot m!}$ then indeed $\mathit{less}_q L(G) \geq \alpha \cdot \log_2 q$.

(iii) From (i) it follows that if $n \geq \frac{1}{d} \cdot \log_2 q$ then

$$f_G(n) \geq 2^{d \cdot n} \geq 2^{d \cdot \frac{1}{d} \cdot \log_2 q} = 2^{\log_2 q} = q.$$

Thus $\mathit{less}_q L(G) \leq \frac{1}{d} \cdot \log_2 q$.

2) Now the rest of the proof of this theorem is carried on in precisely the same way as in the second part of the proof of Theorem 4.

Using Theorems 4 and 5 we can show now that the length sets of DOL languages (even "disguised by Λ-free homomorphisms") code uniquely the information about rank.

Corollary 2. Let G be a DOL system of rank k and let ψ be a Λ-free homomorphism. Let H be a DOL system and let ρ be a Λ-free homomorphism. If $\mathit{Length}\ \psi(L(G)) = \mathit{Length}\ \rho(L(H))$ then H is a DOL system of rank k.

Proof.

First of all from Theorems 4 and 5 it follows that H must be also a DOL system with rank. Let us assume, to the contrary, that H is of rank $\ell \neq k$.

Let $\ell < k$ (the case of $\ell > k$ is proved analogously).

From Theorem 4 it follows that

$$(\exists d_1, d_2)_{N^+} (\forall q)_{N^+} [d_1 \cdot q^{\frac{1}{k}} \leq \mathit{less}_q \psi(L(G)) \leq d_2 \cdot q^{\frac{1}{k}}], \text{ and}$$

$$(\exists \overline{d}_1, \overline{d}_2)_{N^+} (\forall q)_{N^+} [\overline{d}_1 \cdot q^{\frac{1}{\ell}} \leq \mathit{less}_q \rho(L(H)) \leq \overline{d}_2 \cdot q^{\frac{1}{\ell}}].$$

But

$$(\exists q_0)_{N^+}(\forall q)_{N^+}[\textit{if } q > q_0 \textit{ then } d_2 q^{\frac{1}{k}} < \overline{d}_1 \cdot q^{\frac{1}{\ell}}]$$

and so

$$(\forall q)_{N^+}[\textit{if } q > q_0 \textit{ then } \textit{less}_q \psi(L(G)) \leq d_2 \cdot q^{\frac{1}{k}} < \overline{d}_1 \cdot q^{\frac{1}{\ell}} \leq \textit{less}_q \rho(L(H))].$$

Since $\textit{Length } \psi(L(G)) = \textit{Length } \rho(L(H))$, this is a contradiction

Consequently it must be that $\ell = k$ and so the result holds.

We end this paper by summarizing results from [9], [10] and this paper concerning characterizations of polynomially bounded DOL systems. (The notation MR and RME is from [10]).

Theorem 6. Let $G = <\Sigma, \delta, \omega>$ be a DOL system. The following four statements are equivalent.

1) G is polynomially bounded,

2) No useful letter in Σ is expansive,

3) If b is a letter in Σ which is useful and recursive, then b is in $MR \cup RME$.

4) G is a DOL system with rank.

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