ALGORITHMIC PROOFS OF TWO RELATIONS BETWEEN CONNECTIVITY AND THE 1-FACTORS OF A GRAPH

by

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Abstract

An algorithm is used to obtain simple proofs of these two known relations in the theory of matched graphs: A graph with a unique 1-factor contains a matched bridge; an n-connected graph with a 1-factor has at least n totally covered vertices, for $n \geq 2$. The proof of the second result is extended to show some totally covered vertex lies within a distance of 2 from at least n-1 others, for $n \geq 3$.

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1. Introduction

This note is based on an observation of Edmonds that many results about matched graphs can be simply derived by analyzing an algorithm that finds a maximum matching [E]. Using a depth-first version of a cardinality matching algorithm, we derive two known relations between connectivity and 1-factors, giving a stronger version of one of them.

Specifically, we first show a graph with a unique 1-factor has a matched bridge. This result gives the structure of graphs with a unique 1-factor. It was proved by Kotzig, and then by Lovász [L]. Our proof is a simple application of the matching algorithm. Next we show an n-connected graph has at least n totally covered vertices, if $n \geq 2$. This result is useful in analyzing the number of 1-factors of a graph. It was conjectured by Zaks [Z], and proved by Lovász [L], as a consequence of a general theory of the structure of graphs with 1-factors. Our proof gives a slightly stronger result: some totally covered vertex is joined to n-1 other totally covered vertices by paths of length 2, if $n \geq 3$.

2. Definitions

This section gives some relevant definitions from graph theory. For other standard terms, see [H].

A <u>matching</u> M on a graph is a collection of edges, no two of which are incident to the same vertex. If edge vw is in M, it is a <u>matched</u> edge; vertices v and w are <u>matched</u> (to one another). M is a <u>1-factor</u> if every vertex is matched. Throughout this paper, we consider only graphs that have a 1-factor. An <u>alternating path</u> is a path whose edges are alternately matched and unmatched. An <u>alternating cycle</u> C is an alternating path $(v_1, v_2, \ldots, v_{2n}, v_1)$, where all vertices v_i, $1 \le i \le 2n$, are distinct. The set of edges $M \oplus C$† is a 1-factor if M is. Figure 1 shows a 1-factor, {(1,2),(3,4),(5,6)}. Adding edge (1,6) to the graph creates the alternating cycle (1,2,3,4,5,6,1).

A graph is <u>n-connected</u> if it is connected whenever n-1 or fewer vertices are deleted. A <u>bridge</u> is an edge whose removal increases the number of connected components.

Now we introduce some notation. Let $P = (v_1, \ldots, v_n)$ and $Q = (v_{n+1}, \ldots, v_m)$ be paths, and suppose $v_n v_{n+1}$ is an edge. P and Q can be combined into path <u>P·Q</u> $= (v_1, \ldots, v_n, v_{n+1}, \ldots, v_m)$. Reversing the order of vertices in P gives path <u>rev P</u> $= (v_n, \ldots, v_1)$.

3. <u>Depth-First Search of a Graph with a 1-Factor</u>

This section describes a way to search a graph that has a 1-factor. The method is a simple modification of an algorithm that computes a maximum cardinality matching [G1].

The idea is to build a long alternating path, such as (1,2,3,4,5,6) in Figure 1. This is done by starting from a matched edge (edge (1,2)), and repeatedly adding pairs of edges to the end of the path (first (2,3) and (3,4) are added, then (4,5) and (5,6)). The process is complicated by odd length cycles, such as B = (2,3,4,2). For example, if the path (1,2,4,3) is constructed, it cannot be extended from 3. However, (2,3) completes an alternating path from 1 to 4, and this path can be extended (by edges (4,5) and (5,6), as in Figure 2). Cycle B is an example of a blossom, defined below.

We introduce some terminology; the botanical flavor comes from [E]. The subgraph built by a search is called the <u>stem</u>. It starts at

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† $M \oplus C = M \cup C - M \cap C$.

the root vertex; the first edge is matched. An outer vertex in the stem is joined to the root by an alternating path whose first and last edges are matched. If vertex v is outer, P(v) denotes the associated alternating path; if x is a vertex in P(v), then P(v,x) denotes the portion of P(v) from v to x. A vertex in the stem that is not outer is inner. In Figure 2, vertex 1 is the root; 2,3,4 and 6 are outer; P(6) = (6,5,4,3,2,1).

The structure of the stem generalizes an alternating path, in the following sense. The outer vertices can be partitioned into sets, called blossoms, so if the vertices in each blossom are combined (contracted) into a single vertex, the edges remaining in the stem form an alternating path. In this path, if blossom B is incident to the matched edge bc, where $b \in B$ is outer and $c \notin B$ is inner, then c is matched to B; b is the base of blossom B. In Figure 2, the blossoms are {2,3,4} and {6}; the alternating path is formed by edges (1,2),(4,5),(5,6); vertex 1 is matched to blossom {2,3,4}, whose base is 2.

The last blossom in the stem is called the bud. The stem is enlarged by adding edges incident to the bud. In Figure 2, the bud is 6.

Now we give the precise rules for a search. The search starts by making a matched edge rs the stem. Vertex r is the root; s is the bud; s is outer, with alternating path P(s) = (s,r).

The search continues by scanning unmatched edges incident to the bud. Let v be a vertex in the bud, and let vw be an unmatched edge that has not been scanned. Edge vw is scanned as follows:

1. If w is not in the stem, a grow step is done. Let vertex w be matched to vertex z. Edges vw and wz are added to the

stem; z is the new bud; z is outer, with alternating path $P(z) = z \cdot w \cdot P(v)$.

2. If w is an outer vertex not in the bud, a blossom step is done. Let B_i, $1 \le i \le m$, be the blossoms in the portion of the stem from v to w; thus $v \varepsilon B_1$, $w \varepsilon B_m$. Let c_i be the inner vertex matched to B_i. Then edge vw is added to the stem. The bud is a new blossom B, containing vertices c_i, $1 \le i \le m-1$, and B_i, $1 \le i \le m$. The base of B is the vertex matched to c_m. Each new outer vertex c_i, $1 \le i \le m-1$, has alternating path $P(c_i) = \underline{rev}\ P(v,c_i) \cdot P(w)$.

3. If vertex w is the bud or w is inner, no changes are made to the stem.

In Figure 2, the stem is built by doing a grow, blossom, and grow step. If edge (6,3) is added to the graph and is scanned from the bud 6, a blossom step is done. Vertex 5 is made outer, with path P(5) = (5,6,3,4,2,1). The bud becomes {2,3,4,5,6}.

The following simple properties of the stem can be proved by induction on the number of edges scanned. (For a more complete discussion, see [G1,E].) The stem consists of blossoms B_i and inner vertices c_i, for $1 \le i \le n$. For $1 \le i \le n$, B_i is joined to c_i by a matched edge $b_i c_i$, where b_i is the base of B_i. For $2 \le i \le n$, c_i is joined to B_{i-1} by an unmatched edge $c_i d_{i-1}$, where $d_{i-1} \varepsilon B_{i-1}$. If vertex $v \varepsilon B_i$, v is outer and P(v) is an alternating path that starts and ends with matched edges. Further, $P(v) = P(v,b_i) \cdot c_i \cdot P(d_{i-1})$, i.e., P(v) goes from v to the base of B_i, and then to the root. Note this implies P(v) passes through blossoms B_j, for $i \geqq j \geqq 1$ (and also through bases b_j and inner vertices c_j, $i \ge j \ge 1$).

4. Two Relations Between Connectivity and 1-Factors

This section proves two relations between connectivity and 1-factors, by searching a matched graph. The first relation can be used to characterize graphs with a unique 1-factor. It was first proved by Kotzig and then Lovász [L]. Figure 1 illustrates this relation.

Theorem 1: A graph with a unique 1-factor has a bridge that is matched.

Proof: We show that at the end of a search, the matched edge incident to the bud is a bridge.(See Figure 2.)

Let B be the bud at the end of a search. Let vw be an unmatched edge, where $v \in B$. Note $w \in B$: For since no grow step is possible, w is in the stem. Since no blossom step is possible, w is not in a blossom $B' \neq B$. Finally, w is not an inner vertex. For if it is, path $P(v,w) \cdot v$ is an alternating cycle; this implies there are two 1-factors, contrary to assumption. So the only possibility is $w \in B$.

This shows the only edge joining B to a vertex not in B is the matched edge incident to B. This is the desired matched bridge. □

The next relation is used to analyze the number of 1-factors in a graph [G2,Z]. It was conjectured by Zaks [Z], and proved by Lovász [L]. Call a vertex totally covered if every edge incident to it is in some 1-factor. In Figure 3, vertices 1 and 5 are the only totally covered vertices.

Theorem 2: For $n \geq 2$, an n-connected graph with a 1-factor has at least n totally covered vertices.

Proof: Suppose an n-connected graph with a 1-factor has $m < n$ totally covered vertices. We derive a contradiction by showing how to search the graph so a grow or blossom step can always be done, i.e., the search never ends. To do this, we define a set F of outer vertices that always give grow or blossom steps. Specifically, let v be an outer vertex with path $P(v) = (v, v_1, \ldots)$; let x be the first inner vertex in P(v) after v_1 $(x \neq v_1)$. Then

$$F = \{v \mid v \text{ is outer and } P(v,x) \text{ contains an edge that is not in any 1-factor}\}.$$

F has this useful property:

(*) If $v \in F$ is in the bud and edge vw is unmatched, then w is not inner. To show this, suppose on the contrary that w is inner. Then path $P(v,w) \cdot v$ is an alternating cycle, whence every edge in it belongs to a 1-factor. In particular every edge in P(v,x) is in a 1-factor, a contradiction.

Now we give three rules showing how to choose edges to scan in the search. We then show edges can always be chosen according to the rules, so the search never ends.

Initially, the stem consists of the matched edge rs, where the root r is a totally covered vertex (if $m > 0$). Edges for scanning are chosen as follows:

1. Suppose the bud is a non-totally covered vertex v. Scan an edge vw that is not in any 1-factor. This gives a grow or blossom step.
2. Suppose the bud is a totally covered vertex v. If possible, scan an edge vw that gives a blossom step. Otherwise, scan an edge vw that gives a grow step such that the new outer vertex is not totally covered.
3. Suppose the bud is a blossom B, containing more than one vertex. Let b be the base. Scan an edge vw that gives a grow or blossom step, where $v \in B \cap F - b$.

To analyze rules 1-3, we first study the outer vertices not in F. Define N = O-F, where O is the set of all outer vertices. When a vertex becomes outer, it enters set N or F. In a later blossom step, it may move from N to F; however once in F, it remains there.

When the search begins, rs is the stem, and N = {s}. Thereafter, vertices enter N only in rule 2. For suppose z is a new outer vertex. In rule 1, path P(z) contains edge vw; this guarantees $z \in F$. In rule 3, P(z) contains the edge in P(v) that is not any 1-factor; again, $z \in F$.

Note further that in one execution of rule 2, at most one vertex enters N. For suppose edge vw gives a blossom step, and let z be a new outer vertex. Path P(z) contains <u>rev</u> P(v,z). Vertex $v \in F$ (since v is totally covered, it becomes outer in a grow step of rule 1 or rule 3). Thus if z is not matched to v, P(z) contains the edge in P(v) that is not in any 1-factor, and $z \in F$. So only the vertex matched to v can enter N.

Now we show the choices described in rules 1-3 can always be made.

1. In rule 1, v is incident to some edge vw that is not in any 1-factor, by assumption. Vertex w is not inner (as in the proof of (*)). Thus vw gives a grow or blossom step.
2. In rule 2, suppose no edge vw gives a blossom step. Then each edge vw gives a grow step, since w is not inner. To see this, note $v \in F \cup s$, since v is totally covered. If $v \in F$, (*) shows w is not inner; otherwise, v = s, so w is obviously not inner.

   The degree of v is at least n, so there are at least n-1 possible grow steps. There are at most m-1 < n-1 totally covered vertices that can be made outer in a grow step. (The totally covered vertex r cannot be made outer.) So at least one grow step makes a non-totally covered vertex outer.

3. In rule 3, first note $B \cap F - b \neq \phi$. For examining rule 2, it is easy to see that if $z \in B \cap N - b$, then the vertex matched to z is in $B \cap F - b$.

   Now we show

   (1) $|B \cap N \cup b| \leq m$.

   First note $|N| \leq m$, for rule 2 shows each vertex entering N corresponds uniquely to some totally covered vertex. Now consider vertex s; $s \in N$. If $s \notin B$, then $|B \cap N| < m$, so (1) follows. Otherwise if $s \in B$, then $s = b$, and again (1) holds.

   Now consider the graph $G - (B \cap N \cup b)$. Since G is n-connected, this graph is connected, by (1). Thus some vertex $v \in B \cap F - b$ is joined by an edge vw to a vertex $w \notin B$. Vertex w is not inner, by (*). Thus vw gives a grow or blossom step.

Thus an edge can always be chosen according to rules 1-3, and the search never stops. This is the desired contradiction. □

In Figure 3, n=2 and a distance of 4 separates the two totally covered vertices; this distance can be made arbitrarily large by adding more "vertical" edges. We show the opposite holds when $n \geq 3$: a small region of the graph contains n totally covered vertices. Define A(v) as the set of all vertices adjacent to v; $A^2(v)$ is the set of all vertices joined to v by paths of length 2. (Note $v \in A^2(v)$.)

Corollary 1: For $n \geq 3$, an n-connected graph with a 1-factor has a totally covered vertex t, where $A^2(t)$ contains at least n totally covered vertices.

<u>Proof</u>: Let r be a totally covered vertex adjacent to a non-totally covered vertex s. (If no such r exists, all vertices are totally covered, and the Corollary holds.) Choose a 1-factor so rs is matched. Search the graph, starting with r as the root and rs as the first edge of the stem; choose edges according to rules 1-3 of Theorem 2. We prove the Corollary by examining the bud at the end of the search; below we consider the possibilities for the bud corresponding to rules 1-3.

1. The bud is not a non-totally covered vertex, as shown in Theorem 2.
2. Suppose the bud is a totally covered vertex v. No grow or blossom step can be done according to the restrictions of rule 2, since the search has ended. Thus every unmatched edge vw gives a grow step where the new outer vertex is totally covered. At least n-1 such grow steps are possible. So the Corollary holds with t=v.
3. Suppose the bud is a blossom B; let b be the base vertex, and let c be the vertex matched to b. We first show $b \in N$. For suppose the contrary, $b \in F$. This implies $b \neq s$. Thus $c \neq r$, so c is not the only inner vertex. Since G-c is connected, there is an edge vw joining B to some inner vertex w, $w \neq c$. Path $P(v,w) \cdot v$ is an alternating cycle, so every edge in it is in some 1-factor. But P(v,w) contains the edge of P(b) that is not in any 1-factor. This contradiction proves $b \in N$.

   Now we show b=s. For suppose $b \neq s$. Thus b becomes outer in a grow step. In this step, the bud is a totally covered vertex x, since $b \in N$. Clearly $x \neq s$, since s is not totally covered; thus $x \in F$. Let y be the vertex matched to x. Graph

G-c,y is connected. Thus some edge vw joins B to an inner vertex w, $w \neq c,y$. The path $P(v,w) \cdot v$ is an alternating cycle containing the edge of P(x) that is not in any 1-factor. This gives a contradiction, as above. Thus b=s.

So when the search ends, the root r is matched to the bud B. Since G is n-connected, it is easy to see there are n vertices $z \in B$ that are joined to r by an edge. Clearly $z \in N$. For the Corollary, it suffices to show each z is matched to a totally covered vertex. Examining rule 2 we see this is the case if z becomes outer in a blossom step. So it suffices to show z does not become outer in a grow step (of rule 2).

Suppose z becomes outer when the bud is a totally covered vertex v, edge vw is scanned, and a grow step is done. Vertex $v \in F$. So P(v) contains an edge that is not in a 1-factor. P(z) contains the same edge, whence $z \in F$. This contradicts $z \in N$. ☐

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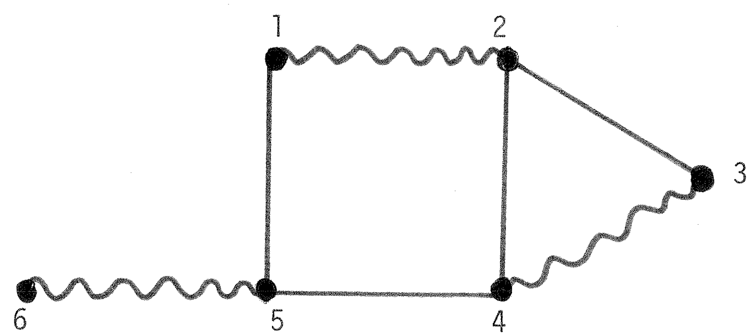

Figure 1: A graph with a 1-factor.

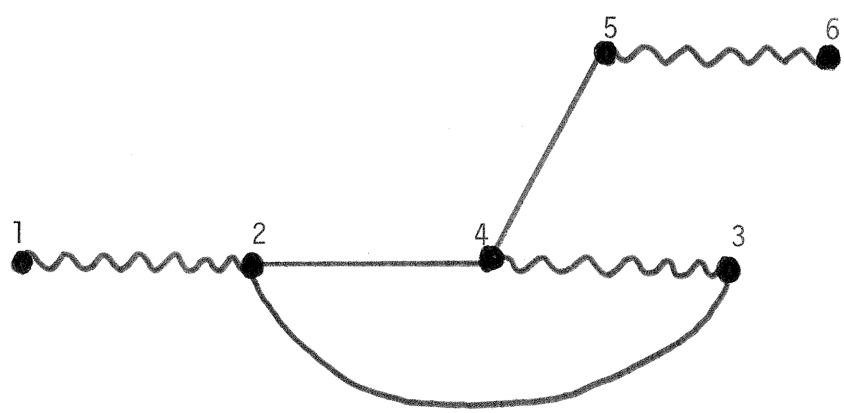

Figure 2: A stem.

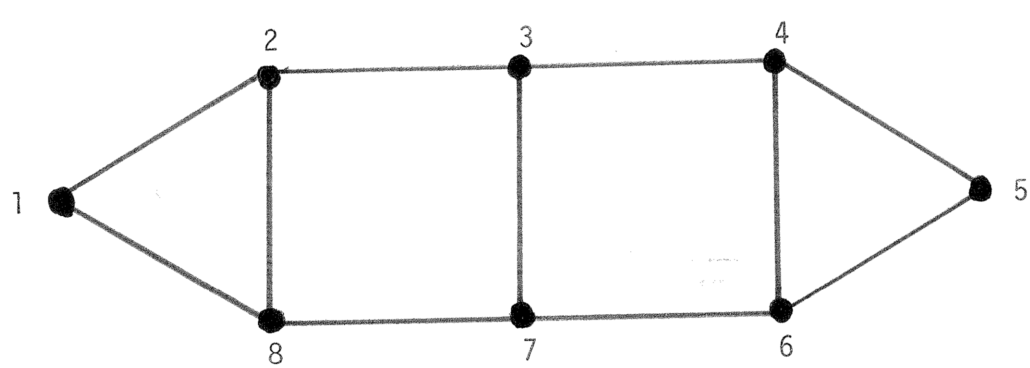

Figure 3: A graph with two totally covered vertices.