

Some Improved Bounds on the Number
of 1-Factors of n -Connected Graphs

by

Harold N. Gabow
Department of Computer Science
University of Colorado
Boulder, Colo. 80309

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1. Introduction

There is an interesting relation between connectivity and the number of distinct 1-factors in a graph. Beineke and Plummer [1] showed that if a graph has a 1-factor and is n -connected, it has n 1-factors. Zaks [3] sharpened the bound on the number of 1-factors to $n(n-2)(n-4)\cdots\cdot 5\cdot 3$ for n odd and $n(n-2)(n-4)\cdots\cdot 4\cdot 2$ for n even. This bound is exact for the complete graphs K_{n+1} (n odd) and the party graph P_6 ($n=4$). We show these are the only graphs that achieve the bound for $n \geq 3$, and improve the bound for the remaining cases.

2. Preliminaries

This section gives some definitions and previous results. For terms not defined here, see [2].

A graph G that has two or more vertices is connected (1-connected) if there is a path between any two vertices. For a positive integer n , G is n -connected if, when any m vertices v_i , $0 \leq m < n$, are deleted, the resulting graph $G - \{v_i | i=1, \dots, m\}$ is connected. (Note an n -connected graph has at least $n+1$ vertices.) G has connectivity n if it is n -connected but not $(n+1)$ -connected. If G is connected but $G - \{v\}$ is not, v separates G .

A 1-factor (perfect matching) of G is a subgraph with exactly one edge incident to every vertex of G . A vertex v is totally covered if every edge vw incident to v is in some 1-factor of G . A fundamental result is the following:

Lemma 1 [3]: A 2-connected graph that has a 1-factor has at least two totally covered vertices.

Define $F(G)$ as the number of distinct 1-factors of G . For any positive integer n , define $f(n)$ as the largest integer such that if G

is an n -connected graph with a 1-factor, then $F(G) \geq f(n)$. Define $g(n)$, a variant of the factorial function, by $g(n) = n$ for $n=1,2$, and $g(n) = ng(n-2)$ for $n \geq 3$. Thus $g(n) = n(n-2) \cdots 5 \cdot 3$ or $n(n-2) \cdots 4 \cdot 2$, depending on whether n is odd or even. An induction using Lemma 1 shows $f(n) \geq g(n)$ for $n \geq 1$.

Define $f^*(n)$ as the second-lowest possible number of 1-factors for an n -connected graph. That is, $f^*(n)$ is the largest integer such that if G is an n -connected graph with $F(G) > f(n)$, then $F(G) \geq f^*(n)$.

The complete graph K_n consists of n vertices and all possible edges between them. If n is odd, K_{n+1} is n -connected and has a 1-factor. In fact, $F(K_{n+1}) = g(n)$; so $f(n) = g(n)$ for n odd.

If n is even, the party graph P_n is K_n with the edges of a 1-factor deleted. P_n is $(n-2)$ -connected. It is easy to check $F(P_4) = 2$, $F(P_6) = 8$, and $F(P_{n+2}) = n(F(P_n) + F(P_{n-2}))$ for $n \geq 6$.

3. The New Bounds

We begin by analyzing 3- and 4- connected graphs, confirming conjectures of Zaks on $f^*(3)$ and $f^*(4)$.

Lemma 2: $f(3) = 3$; K_4 is the only 3-connected graph G with $F(G) = 3$;
 $f^*(3) = 4$.

Proof: Since $F(K_4) = 3$, we have $f(3) \leq 3$. Similarly, $f^*(3) \leq 4$ is shown by the graph consisting of a cycle on six vertices, $v_1v_2v_3v_4v_5v_6$, plus the edges v_1v_4 , v_2v_6 , v_3v_5 . Lemma 1 implies $f(3) \geq 3$, so $f(3) = 3$.

It remains only to show that if $G \neq K_4$ is a 3-connected graph with a 1-factor, then $F(G) \geq 4$. We assume $F(G) = 3$ and derive a contradiction.

G contains a totally covered vertex t , by Lemma 1. The assumptions imply t is adjacent to exactly three vertices v_i , $i=1,2,3$. Further, each graph $G - \{t, v_i\}$ has exactly one 1-factor, i.e., $F(G - \{t, v_i\}) = 1$.

Using Lemma 1, we see $G - \{t, v_i\}$ is not 2-connected; however it is 1-connected. Thus there is a vertex c_i separating $G - \{t, v_i\}$ i.e., $G - \{t, v_i, c_i\}$ is not connected. Note $c_i \neq v_j$ for any i, j , since $G - \{t, v_i, v_j\}$ is connected. (The assumption $G \neq K_4$ is used here.)

Let v_i, v_j, v_k be the vertices adjacent to t , in some order. Graph $G - \{v_i, c_i\}$ is separated by t . Thus v_j and v_k are in different components of $G - \{t, v_i, c_i\}$.

The proof is completed by contradicting this fact, i.e., we find a path between v_j and v_k in $G - \{t, v_i, c_i\}$. We do this by finding paths between v_j and c_k , v_k and c_j , and c_j and c_k .

Graph $G - \{t, c_i\}$ is separated by v_i . So there is a simple path P between v_j and v_k , containing v_i . The part of P between v_j and v_i is in $G - \{t, v_k\}$; it contains c_k , since v_j and v_i are in different components of $G - \{t, v_k, c_k\}$. This gives the desired path between v_j and c_k . Similarly the part of P between v_k and v_i gives the desired path between v_k and c_j . Note further, vertices c_i, c_j , and c_k are distinct.

Now we find the desired path between c_j and c_k . Since $G - \{t, v_i, v_j\}$ is connected, it contains a path Q between c_j and c_k , and also a path between c_j and c_i . Assume without loss of generality that Q does not contain c_i . (If it does, interchange indices i and j .) So Q is in $G - \{t, v_i, c_i\}$, and is the last desired path. QED

Lemma 3: $f(4) = 8$; P_6 is the only 4-connected graph G with $F(G) = 8$; $f^*(4) = 10$.

Proof: We see $f(4) \leq 8$ and $f^*(4) \leq 10$ by counting the number of 1-factors in P_6 and $P_6 + e$, where the latter graph is P_6 plus one extra edge.

Let G be a 4-connected graph with a 1-factor. Let t be a totally covered vertex and let tv be an edge. Then $G - \{t, v\}$ is 2-connected, so $F(G - \{t, v\}) \geq 2$. Since t has degree at least four, we see $F(G) \geq 8$. Thus $f(4) = 8$.

It remains only to show that $F(G) \geq 10$ if $G \neq P_6$. The argument divides into two cases.

Case 1: There is a totally covered vertex t and an edge tv , such that $G - \{t, v\}$ is 3-connected.

Apply the results of Lemma 2 to $G - \{t, v\}$. If $G - \{t, v\} \neq K_4$, then $F(G - \{t, v\}) \geq 4$, so $F(G) \geq 4 + 3 \cdot 2 = 10$, as desired. Otherwise if $G - \{t, v\} = K_4$, graph G has six vertices and contains at least the edges of $P_6 + e$. Thus $F(G) \geq F(P_6 + e) = 10$.

Case 2: For every totally covered vertex t and every edge tv , $G - \{t, v\}$ has connectivity 2.

If a totally covered vertex has degree five or more, then $F(G) \geq 5 \cdot 2 = 10$, as desired. Thus assume all totally covered vertices have degree four. Assume further that $F(G) < 10$. Below we deduce $G = P_6$.

Let t be a totally covered vertex, adjacent to vertices v_i , $i=1,2,3,4$. We first show vertices v_i form a cycle on four vertices, with no other edges joining them.

No three vertices v_i form a cycle. For suppose $v_1 v_2 v_3$ is a cycle. Then since $G - \{v_4\}$ is 3-connected, $G - \{t, v_4\}$ is 3-connected. But this violates the assumption of Case 2.

Since $F(G) < 10$, assume without loss of generality that $F(G - \{t, v_i\}) = 2$, for $i=1,2,3$. Since $G - \{t, v_i\}$ is 2-connected, it has two totally covered vertices. Both have degree two in $G - \{t, v_i\}$. Any vertex has degree at least four in G . So both totally covered vertices of $G - \{t, v_i\}$ are adjacent to t and v_i . In particular, both are among the vertices v_j .

Thus each vertex v_i , $i=1,2,3$, is adjacent to two other vertices v_j . Since no three vertices v_j form a cycle, it is easy to see the four vertices v_j form a cycle, with no other edges.

Now we find other totally covered vertices besides t . Without loss of generality, assume the cycle found above is $v_1v_2v_3v_4$. Vertex v_2 is totally covered in $G - \{t, v_i\}$, for $i=1,3$. Thus v_2 is totally covered in G . This in turn implies v_1 is totally covered in G .

The above argument shows the totally covered vertex v_2 is adjacent to four vertices joined in a cycle. This cycle is v_1tv_3s , where s is a vertex in G . Similarly, v_1 is adjacent to four vertices joined in a cycle, v_2tv_4s . Thus the six vertices $s, t, v_i, i=1,2,3,4$, and their interconnecting edges, form the party graph P_6 .

G contains no other vertices, since $G - \{s, v_3, v_4\}$ is connected. Thus $G = P_6$. QED

Now we extend these results to higher connected graphs. Zaks conjectured that K_{n+1} (n odd) and P_6 ($n=4$) are the only graphs with exactly $g(n)$ 1-factors, for $n \geq 3$. Theorems 4 and 5 confirm this.

Theorem 4: Let $n \geq 3$ be odd. Then $f(n) = g(n)$; K_{n+1} is the only n -connected graph G with $F(G) = g(n)$; $f^*(n) \geq \frac{4}{3} g(n)$.

Proof: We show by induction on odd n that if $G \neq K_{n+1}$ is an n -connected graph with a 1-factor, then $F(G) \geq \frac{4}{3} g(n)$. Lemma 2 establishes the base case, $n=3$. So assume the assertion holds for some $n \geq 3$. Let $G \neq K_{n+3}$ be an $(n+2)$ -connected graph with a 1-factor. Let t be a totally covered vertex, and let tv be an edge.

Graph $G - \{t, v\}$ is n -connected and has a 1-factor. Further, it is not K_{n+1} . For otherwise, each vertex w in $G - \{t, v\}$ is adjacent to both t and v , since w has degree at least $n+2$ in G . This implies $G = K_{n+3}$, contrary to assumption.

The inductive assertion shows $F(G - \{t, v\}) \geq \frac{4}{3} g(n)$. Since there are at least $n+2$ vertices v , $F(G) \geq \frac{4}{3} g(n+2)$. QED

Theorem 5: Let $n \geq 6$ be even. Then $f(n) \geq \frac{5}{4} g(n)$.

Proof: We first show for $n=6$, $f(6) \geq \frac{5}{4} g(6) = 60$. Let G be a 6-connected graph with a 1-factor. It is easy to see a totally covered vertex t has at least six edges tv such that $G - \{t, v\} \neq P_6$. So Lemma 3 shows $F(G) \geq 6 \cdot 10 = 60$, whence $f(6) \geq 60$.

The general case now follows by induction, with $n=6$ as the base. QED

Theorem 5 gives an exact bound for $n=6$, as shown by the party graph P_8 . We conjecture that for any even $n \geq 2$, $f(n) = F(P_{n+2})$.

References

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