

ON DECOMPOSING SOME ETOL LANGUAGES INTO  
DETERMINISTIC ETOL LANGUAGES<sup>\*</sup>

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## ABSTRACT

This paper provides a method of decomposing a subclass of ETOL languages into deterministic ETOL languages. This allows one to use every known example of a language which is not a deterministic ETOL language to produce languages which are not ETOL languages.

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## I. INTRODUCTION

The theory of L systems originated from the work of A. Lindenmayer (see Lindenmayer [ii]). Although initially proposed as a theory for the development of filamentous organisms, in the last four years it turned out to be useful and interesting from both the biological and formal points of view (see, e.g., Herman and Rozenberg [g ], and Rozenberg and Salomaa [13]).

One of the central families of L languages (that is languages generated by L systems) is the family of ETOL languages (see, e.g. Downey [2 ], Rozenberg [12] and Salomaa [15]). An important research area in the theory of ETOL systems and languages is to provide results which would facilitate proofs that certain languages are not ETOL languages. Although some such results are already available (see, e.g., Ehrenfeucht and Rozenberg [4 ], and Ehrenfeucht and Rozenberg [5 ]), a lot of work in this direction remains to be done.

This paper provides a criterion for proving that some languages are not ETOL languages. In fact it shows how, in certain cases, to reduce this problem to proving that some languages are not deterministic ETOL languages (see Rozenberg [12] and Ehrenfeucht and Rozenberg [6 ]). This is a great help indeed, because it is easier to investigate the structure of derivations in a deterministic ETOL system, and quite a number of examples of languages that are not deterministic ETOL languages are already available (see, e.g., Ehrenfeucht and Rozenberg [7 ] and Ehrenfeucht and Rozenberg [8 ]).

As a corollary of our results we get that the family of ETOL languages is strictly included in the family of index languages of Aho (see, Aho [1 ]). This was quite an important open problem of a rather long standing (see, e.g., Downey [2 ], Salomaa [15] and Salomaa [16]).

We assume the reader to be familiar with rudiments of formal language theory, e.g. in the scope of the first four chapters of Hopcroft and Ullman [10].

## II. DEFINITIONS

In this section we provide definitions and examples of systems and languages used in this paper.

Definition 1. An extended table L system without interactions, abbreviated as an ETOL system, is defined as a four-tuple  $G = \langle V, P, \omega, \Sigma \rangle$  such that:

(1)  $V$  is a finite set (called the alphabet of  $G$ ),

(2)  $P$  is a finite set (called the set of tables of  $G$ ),  $P = \{P_1, \dots, P_f\}$

for some  $f \geq 1$ , each element of which is a finite subset of  $V \times V^*$ .  $P$  satisfies the following (completeness) condition:

$$(\forall P) \ (\forall a) \ (\exists \alpha) \ (\exists \alpha') \ (\langle a, \alpha \rangle \in P),$$

(3)  $\omega \in V^+$  (called the axiom of  $G$ ),

(4)  $\Sigma \subseteq V$  (called the target alphabet of  $G$ ).

We assume that  $V$ ,  $\Sigma$ , and each  $P$  in  $P$  are nonempty sets.

Definition 2. Let  $G = \langle V, P, \omega, \Sigma \rangle$  be an ETOL system. Let  $x \in V^+$ ,  $x = a_1 \dots a_k$ , where each  $a_j$ ,  $1 \leq j \leq k$ , is an element of  $V$ , and let  $y \in V^*$ . We say that  $x$  directly derives  $y$  in  $G$  (denoted  $x \xrightarrow{G} y$ ) if and only if there exist  $P$  in  $P$  and  $p_1, \dots, p_k$  in  $P$  such that  $p_1 = \langle a_1, \alpha_1 \rangle$ ,  $p_2 = \langle a_2, \alpha_2 \rangle$ ,  $\dots$ ,  $p_k = \langle a_k, \alpha_k \rangle$  (for some  $\alpha_1, \dots, \alpha_k \in V^*$ ) and  $y = \alpha_1 \dots \alpha_k$ . We say that  $x$  derives  $y$  in  $G$  (denoted  $x \xrightarrow{*G} y$ ) if and only if either (i) there exists a sequence of words  $x_0, x_1, \dots, x_n$  in  $V^*$  (with  $n > 1$ ) such that  $x_0 = x$ ,  $x_n = y$  and  $x_0 \xrightarrow{G} x_1 \xrightarrow{G} x_2 \dots \xrightarrow{G} x_n$ ; or (ii)  $x = y$ .

Definition 3. Let  $G = \langle V, P, \omega, \Sigma \rangle$  be an ETOL system. The language of  $G$  (denoted as  $L(G)$ ) is defined as  $L(G) = \{x \in \Sigma^* : \omega \xrightarrow{*G} x\}$ .

Definition 4. An ETOL system  $G = \langle V, P, \omega, \Sigma \rangle$  is called deterministic (abbreviated EDTOL system) if for each  $P$  in  $P$  and each  $a$  in  $V$  there exists exactly one  $\alpha$  in  $V^*$  such that  $\langle a, \alpha \rangle \in P$ .

Definition 5. Let  $\Sigma$  be a finite alphabet and  $K \subseteq \Sigma^*$ .  $K$  is called an ETOL (EDTOL) language if and only if there exists an ETOL (EDTOL) system  $G$  such that  $L(G) = K$ .

We shall use  $L(\text{ETOL})$  and  $L(\text{EDTOL})$  to denote the class of ETOL languages and the class of EDTOL languages, respectively.

Definition 6. An ETOL system  $G = \langle V, P, \omega, \Sigma \rangle$  is called synchronized if for every  $x, y$  such that  $x \in V^* \Sigma V^*$ ,  $y \in V^+$  and  $x$  derives  $y$  in at least one step, then  $y \in V^*(V - \Sigma) V^*$ .

Definition 7. Let  $G = \langle V, P, \omega, \Sigma \rangle$  be an ETOL system with  $P = \{P_1, \dots, P_f\}$ . Let  $Z \subseteq V$ . For  $P$  in  $\mathcal{P}$  a  $P(Z)$  table is a set of ordered pairs  $\langle a, \alpha \rangle$ ,  $a$  in  $V$  and  $\alpha$  in  $V^*$ , such that, for each  $a$  in  $(V - Z)$  each element  $\langle a, \alpha \rangle$  from  $P$  is in  $P(Z)$ , for each  $a$  in  $Z$ ,  $P(Z)$  contains exactly one element  $\langle a, \alpha \rangle$  from  $P$  and  $P(Z)$  contains nothing else. An ETOL system  $H = \langle V, \bar{P}, \omega, \Sigma \rangle$  is called the Z-combinatorially complete version of G if  $\bar{P} = \{T : \text{for some } P \text{ in } \mathcal{P}, T \text{ is a } P(Z) \text{ table}\}$ ; If  $Z = V$ , then we say that it is the combinatorially complete version of G.

Notation. Let  $G = \langle V, P, \omega, \Sigma \rangle$  be an ETOL system. If  $\langle a, \alpha \rangle$  is an element of some  $P$  in  $\mathcal{P}$ , then we call it a production (for a in P) and write  $a \xrightarrow{P} \alpha$ . A derivation in  $G$  is a sequence of words  $(x_0, x_1, \dots, x_n)$  such that  $x_0 = \omega$  and  $x_j \xrightarrow[G]{} x_{j+1}$ , for  $0 \leq j \leq n-1$ . (We also say that it is a derivation of  $x_n$  in G). Sometimes by a derivation we shall mean a sequence  $(x_0, \dots, x_n)$  together with the precise set of productions used in each derivation step but this will always be clear from the context and should not lead to confusion.

Example 1. Let  $G_1 = \langle \{a, b, A, B, C, D, F\}, P, CD, \{a, b\} \rangle$  where  $P = \{P_1, P_2, P_3\}$  and  
 $P_1 = \{a \rightarrow F, b \rightarrow F, A \rightarrow A, B \rightarrow B, C \rightarrow ACB, D \rightarrow DA\}$ ,  
 $P_2 = \{a \rightarrow F, b \rightarrow F, A \rightarrow A, B \rightarrow B, C \rightarrow CB, D \rightarrow D\}$ ,

$$P_3 = \{a \rightarrow F, b \rightarrow F, A \rightarrow a, B \rightarrow b, C \rightarrow \Lambda, D \rightarrow \Lambda\}.$$

$G_1$  is a synchronized EDTOL system and

$$L(G_1) = \{a^n b^m a^n : n \geq 0, m \geq n\}.$$

Example 2. Let  $G_2 = \langle \{a, b, A, A', B, B', C, C', F\}, P, A B C, \{a, b\} \rangle$

where  $P = \{P\}$  and

$$P = \{a \rightarrow F, b \rightarrow F, c \rightarrow F, A \rightarrow A'A, A \rightarrow a, B \rightarrow B'B, B \rightarrow b, C \rightarrow C'C,$$

$$C \rightarrow c, A' \rightarrow A', A' \rightarrow a, B' \rightarrow B', B' \rightarrow b, C' \rightarrow C', C' \rightarrow c, F \rightarrow F\}.$$

$G_2$  is a synchronized (but not deterministic) ETOL system and  $L(G) = \{a^n b^n c^n : n \geq 1\}.$

III. RESULTS

Theorem 1. Let  $\Sigma_1, \Sigma_2$  be two finite disjoint alphabets and let  $K_1 \subseteq \Sigma^*$ ,  $K_2 \subseteq \Sigma^*$ . Let  $f$  be a bijective function from  $K_1$  onto  $K_2$ . Let  $K = \{wf(w) : w \in K_1\}$ . If  $K \in L(ETOL)$  then  $K, K_1, K_2 \in L(EDTOL)$ .

Proof

The idea of our proof is to start with an arbitrary ETOL system generating  $K$  and then to construct EDTOL systems generating  $K, K_1$  and  $K_2$  respectively. As the construction is quite involved we have split it up into several steps.

Let  $K$  satisfy the statement of the theorem.

Let  $G = \langle V, P, \omega, \Sigma \rangle$ , where  $\Sigma = \Sigma_1 \cup \Sigma_2$ , be a synchronized ETOL system generating  $K$ . By Theorems 3 and 4 in Rozenberg [12], we may assume that  $\omega = S$ , where  $S \in V - \Sigma$ , and, for every  $a$  in  $V$  and every  $P$  in  $P$ , if  $a \xrightarrow{P} \alpha$  then either  $\alpha \in \Sigma^*$  or  $\alpha \in (V - \Sigma)^+$ .

Now we present, in several steps, our construction.

STEP 1.

Let  $V^{(1)} = \{a^{(1)} : a \in V - \Sigma_2\}$ ,  $V^{(2)} = \{a^{(2)} : a \in V - \Sigma_1\}$

and  $V^{(m)} = \{a^{(m)} : a \in V - \Sigma\}$ . If  $j \in \{1, 2, m\}$ , and  $\alpha = b_1 \dots b_k$  with  $k \geq 1$  and  $b_1, \dots, b_k \in V$ , then  $\alpha^{(j)} = b_1^{(j)} \dots b_k^{(j)}$ . Also for  $j \in \{1, 2, m\}$ ,  $\Lambda^{(j)} = \Lambda$ . Let

$\Sigma_1^{(1)} = \{a^{(1)} : a \in \Sigma_1\}$  and  $\Sigma_2^{(2)} = \{a^{(2)} : a \in \Sigma_2\}$ . Let  $F$  be a new symbol ( $F \notin V \cup V^{(1)} \cup V^{(2)} \cup V^{(m)} \cup \Sigma$ ). Let  $V_1 = V^{(1)} \cup V^{(2)} \cup V^{(m)} \cup \Sigma \cup \{F\}$ .

For each table  $P$  in  $P$  we construct a new table  $\bar{P}$  as follows:

- (i) if  $a \xrightarrow{P} \alpha$  with  $\alpha \in \Sigma_1^*$ , then  $a^{(1)} \xrightarrow{\bar{P}} \alpha^{(1)}$ ,
- (ii) if  $a \xrightarrow{P} \alpha$  with  $\alpha \in \Sigma_2^*$ , then  $a^{(2)} \xrightarrow{\bar{P}} \alpha^{(2)}$ ,
- (iii) if  $a \xrightarrow{P} \alpha$  with  $\alpha \in (V - \Sigma)^+$ , say  $\alpha = b_1 \dots b_k$  for some  $k \geq 1$  and

$b_1, \dots, b_k \in (V - \Sigma)$ , then

$$a^{(1)} \xrightarrow{\bar{P}} \alpha^{(1)},$$

$$a^{(2)} \xrightarrow{\bar{P}} \alpha^{(2)},$$



$$a^{(m)} \xrightarrow{\bar{P}} \alpha^{(1)},$$

$$a^{(m)} \xrightarrow{\bar{P}} b_1^{(1)} \dots b_{\ell-1}^{(1)} b_{\ell}^{(m)} b_{\ell+1}^{(2)} \dots b_k^{(2)} \text{ for every } \ell \text{ in } \{1, \dots, k\},$$

$$a^{(m)} \xrightarrow{\bar{P}} b^{(1)} \dots b_{\ell-1}^{(1)} b_{\ell}^{(2)} \dots b_k^{(2)} \text{ for every } \ell \text{ in } \{1, \dots, k\},$$

$$(iv) F \xrightarrow{\bar{P}} F,$$

$$(v) \text{ for every } j \text{ in } \{1, 2, m\} \text{ and every } a \text{ in } V, a^{(j)} \xrightarrow{\bar{P}} F,$$

(this is the easiest way to have  $\bar{P}$  satisfying the completeness condition),

(vi) only productions obtained from (i) through (v) are in  $\bar{P}$ .

Let  $P_t$  be a new table such that

$$P_t = \{a^{(1)} \rightarrow a : a^{(1)} \in \Sigma_1^{(1)}\} \cup \{a^{(2)} \rightarrow a : a^{(2)} \in \Sigma_2^{(2)}\} \cup \{a \rightarrow F : a \in V - (\Sigma_1^{(1)} \cup \Sigma_2^{(2)})\}.$$

$$\text{Let } P_1 = \{P_t\} \cup \{\bar{P} : P \in \mathcal{P}\}.$$

$$\text{Finally let } G_1 = \langle V_1, P_1, S^{(m)}, \Sigma \rangle.$$

### STEP 2.

Let  $G_2 = \langle V_2, P_2, S^{(m)}, \Sigma \rangle$  be the  $V^{(m)}$ -combinatorially complete version of  $G_1$ . (Note that  $V_2 = V_1$ ).

### STEP 3.

$$\text{Let } V^{(m:1)} = \{a^{(m:1)} : a^{(m)} \in V^{(m)}\} \text{ and } V^{(m:2)} = \{a^{(m:2)} : a^{(m)} \in V^{(m)}\}.$$

$$\text{Let } V_3 = (V - V^{(m)}) \cup V^{(m:1)} \cup V^{(m:2)}.$$

For each table  $P$  in  $\mathcal{P}$  we construct a new table  $\hat{P}$  as follows:

$$(i) \text{ if } a \in V - V^{(m)} \text{ and } a \xrightarrow{P} \alpha, \text{ then } a \xrightarrow{\hat{P}} \alpha,$$

$$(ii) \text{ if } a^{(m)} \in V^{(m)} \text{ and } a^{(m)} \xrightarrow{P} \alpha^{(1)} b^{(m)} \alpha^{(2)} \text{ for some } b^{(m)} \text{ in } V^{(m)}, \text{ then}$$

$$a^{(m:1)} \xrightarrow{\hat{P}} \alpha^{(1)} b^{(m:1)} \text{ and } a^{(m:2)} \xrightarrow{\hat{P}} b^{(m:2)} \alpha^{(2)},$$

$$(iii) \text{ if } a^{(m)} \in V^{(m)} \text{ and } a^{(m)} \xrightarrow{P} \alpha^{(1)}, \text{ then}$$

$$a^{(m:1)} \xrightarrow{\hat{P}} \alpha^{(1)} \text{ and } a^{(m:2)} \xrightarrow{\hat{P}} \alpha^{(2)},$$

(iv) if  $a^{(m)} \in V^{(m)}$  and  $a^{(m)} \xrightarrow{P} \alpha^{(2)}$ , then

$$a^{(m:1)} \xrightarrow{\hat{P}} \Lambda \text{ and } a^{(m:2)} \xrightarrow{\hat{P}} \alpha^{(2)}$$

(v) if  $a^{(m)} \in V^{(m)}$  and  $a^{(m)} \xrightarrow{P} \alpha^{(1)} \alpha^{(2)}$ , then

$$a^{(m:1)} \xrightarrow{\hat{P}} \alpha^{(1)} \text{ and } a^{(m:2)} \xrightarrow{\hat{P}} \alpha^{(2)}.$$

(vi) only productions obtained from (i) through (v) are in  $\hat{P}$ .

$$\text{Let } P_3 = \{\hat{P} : P \in P_2\}.$$

$$\text{Finally let } G_3 = \langle V_3, P_3, S^{(m:1)} S^{(m:2)}, \Sigma \rangle$$

STEP 4.

Let  $G_4 = \langle V_4, P_4, S^{(m:1)} S^{(m:2)}, \Sigma \rangle$  be the combinatorially complete version of  $G_3$ .

$$\text{Let } \bar{P}_t = \{a^{(1)} \rightarrow a : a^{(1)} \in \Sigma_1^{(1)}\} \cup \{a^{(2)} \rightarrow \Lambda : a^{(2)} \in \Sigma_2^{(2)}\} \cup \{a \rightarrow F : a \in V_4 - (\Sigma_1^{(1)} \cup \Sigma_1^{(2)})\}.$$

$$\text{Let } \bar{\bar{P}}_t = \{a^{(1)} \rightarrow \Lambda : a^{(1)} \in \Sigma^{(1)}\} \cup \{a^{(2)} \rightarrow a : a^{(2)} \in \Sigma_2^{(2)}\} \cup \{a \rightarrow F : a \in V_4 - (\Sigma_1^{(1)} \cup \Sigma_2^{(2)})\}.$$

$$\text{Let } \bar{P}_4 = (P_4 - \{P_t\}) \cup \{\bar{P}_t\} \text{ and } \bar{\bar{P}}_4 = (P_4 - \{P_t\}) \cup \{\bar{\bar{P}}_t\}.$$

$$\text{Let } G_4^{(1)} = \langle V_4, \bar{P}_4, S^{(m:1)} S^{(m:2)}, \Sigma \rangle \text{ and } G_4^{(2)} = \langle V_4, \bar{\bar{P}}_4, S^{(m:1)} S^{(m:2)}, \Sigma \rangle.$$

To complete the proof of Theorem 1 we have to show that  $L(G_4) = K$ ,  $L(G_4^{(1)}) = K_1$  and  $L(G_4^{(2)}) = K_2$ .

Let us first prove that  $L(G_4) = K$ .

We do this by proving that the sequence of ETOL systems  $G, G_1, G_2, G_3, G_4$  has this property that they all generate the same language.

$$I) L(G) = L(G_1).$$

We shall present now the main idea behind the proof of this equality, leaving to the reader the formal proof.

$$L(G) \subseteq L(G_1).$$

If one takes a derivation tree in  $G$  of a word  $x$  of the form  $x_1 x_2$  with  $x_1 \in \Sigma_1^*$  and  $x_2 \in \Sigma_2^*$ , then (in the bottom-up fashion) one can classify all the nodes in this tree into three categories: those which contribute to  $x_1$ , those which contribute to  $x_2$  and those which contribute to both  $x_1$  and  $x_2$ . If a node belongs to the first category and its label is  $a$  then we change it to  $a^{(1)}$ , if a node belongs to the second category and its label is  $a$  then we change it to  $a^{(2)}$ , and if a node belongs to the third category and its label is  $a$  then we change it to  $a^{(m)}$ . But it clearly follows from the construction of  $G$  that such a derivation tree with one extra level added (corresponding to the application of  $P_t$  from  $P_1$ ) corresponds to a derivation in  $G_1$  and consequently  $x$  is in  $L(G_1)$ . Thus  $L(G) \subseteq L(G_1)$ .

$$L(G_1) \subseteq L(G).$$

If one takes a derivation of a word  $x$  in  $L(G_1)$  and then omits all superscripts in the letters of the form  $a^{(i)}$  with  $i$  in  $\{1,2,m\}$  and also omits the last level (corresponding to the application of table  $P_t$  from  $P_1$ ), then, clearly, one gets a valid derivation in  $L(G)$ . Consequently  $x$  is in  $L(G)$ . Thus  $L(G) \subseteq L(G_1)$ .

$$\text{II). } L(G_1) = L(G_2).$$

This equality follows immediately by observing that in each derivation in  $G_1$  each intermediate word contains at most one occurrence of a letter of the form  $a^{(m)}$ .

$$\text{III). } L(G_2) = L(G_3).$$

What  $G_3$  does is simply split each derivation tree of a word  $x$  in  $L(G_2)$  into two trees (glued together): one corresponding to the derivation of the prefix  $x_1$  (in  $\Sigma_1^*$ ) of  $x$  and the second corresponding to the derivation of the suffix  $x_2$  (in  $\Sigma_2^*$ ) of  $x$ .

$$\text{IV). } L(G_3) = L(G_4).$$

$$\text{Clearly } L(G_4) \subseteq L(G_3).$$

Now let  $D = (y_0 = S^{(m:1)} S^{(m:2)}, y_1, y_2, \dots, y_n)$  be a derivation in  $G_3$  of the

word  $y_n$  which is of the form  $\beta_1\beta_2$  where  $\beta_1 \in \Sigma_1^*$  and  $\beta_2 \in \Sigma_2^*$ . Let  $T_1, \dots, T_n$  be the sequence of tables from  $P_3$  that was applied in this particular derivation.

Let us change derivation  $D$  (in a top-down fashion) to derivation  $\bar{D} = (\bar{y}_0 = s^{(m:1)} s^{(m:2)}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_n = \bar{\beta}_1\bar{\beta}_2)$ , with  $\bar{\beta}_1$  in  $\Sigma_1^*$ , as follows:

- 1) for every  $j$  in  $\{1, \dots, n\}$  and every  $a$  in  $V^{(1)} \cup V^{(m:1)}$  we rewrite each occurrence of  $a$  in  $\bar{y}_{j-1}$  by the same production from  $T_j$ , but it must be a production used in rewriting an occurrence of  $a$  in  $y_{j-1}$ ,
- 2) for every  $j$  in  $\{1, \dots, n\}$  and every  $a$  in  $V - (V^{(1)} \cup V^{(m:1)})$  we rewrite each occurrence of  $a$  in  $\bar{y}_{j-1}$  in exactly the same way it was rewritten in  $y_{j-1}$ .

We can note now that, since  $f$  is a bijective function,  $\bar{\beta}_1 = \beta_1$ .

Let us change derivation  $D$  (in a top-down fashion) to derivation  $\bar{\bar{D}} = (\bar{\bar{y}}_0 = s^{(m:1)} s^{(m:2)}, \bar{\bar{y}}_1, \dots, \bar{\bar{y}}_n = \beta_1\bar{\bar{\beta}}_2)$ , with  $\bar{\bar{\beta}}_2$  in  $\Sigma_2^*$  as follows:

- 3) for every  $j$  in  $\{1, \dots, n\}$  and every  $a$  in  $V^{(2)} \cup V^{(m:2)}$  we rewrite each occurrence of  $a$  in  $\bar{\bar{y}}_{j-1}$  by the same production from  $T_j$ , but it must be a production used in rewriting an occurrence of  $a$  in  $y_{j-1}$ ,
- 4) for every  $j$  in  $\{1, \dots, n\}$  and every  $a$  in  $V - (V^{(2)} \cup V^{(m:2)})$  we rewrite each occurrence of  $a$  in  $\bar{\bar{y}}_{j-1}$  in exactly the same way it was rewritten in  $y_{j-2}$ :

We can note now that, since  $f$  is a bijective function,  $\bar{\bar{\beta}}_2 = \beta_2$ .

It follows immediately from our conclusions about  $\bar{D}$  and  $\bar{\bar{D}}$  that there exists a derivation of  $y_n$  in  $G_4$ . Thus  $L(G_3) \subseteq L(G_4)$ .

From I) through IV) we get that  $L(G) = L(G_4)$ .

But  $G_4$  is an EDTOL system and consequently  $K$  is in  $L(\text{EDTOL})$ .

We leave to the reader the obvious proofs that  $K_1 = L(G_4^{(1)})$  and  $K_2 = L(G_4^{(2)})$ .

Both of these equalities follow easily from the observation that  $G_4$  is a synchronized EDTOL system and in every successful derivation in  $G_4$  the last table applied must be  $P_t$ . But  $G_4^{(1)}$  and  $G_4^{(2)}$  are EDTOL systems and consequently both  $K_1$  and  $K_2$  are in  $L(\text{EDTOL})$ .

This completes the proof of Theorem 1.

IV. APPLICATIONS

First of all, using Theorem 1 and Theorem 2 we can provide examples of languages which are not in  $L(ETOL)$ .

Let  $W_1 = \{x \in \{0, 1\}^+ : |x| = 2^n \text{ for some } n \geq 0\}$ .

Let us recall the following result proved in Ehrenfeucht and Rozenberg [ 3 ] and Ehrenfeucht and Rozenberg [ 7 ].

Lemma 1.  $W_1 \in L(EDTOL)$ .

Let  $f_1$  be a function from  $\{0, 1\}^+$  into  $\{c, d\}^+$  defined as follows. For  $k \geq 1$  and  $b_1, \dots, b_k \in \{0, 1\}$ ,  $f_1(b_1 \dots b_k) = x_1 x_2 \dots x_k$  where for every  $i$  in  $\{1, \dots, k\}$

$$x_i = \begin{cases} c & \text{if } b_i = 0, \\ d & \text{if } b_i = 1. \end{cases}$$

Let  $W_2 = \{x f_1(x) : x \in W_1\}$ .

Proposition 1.  $W_2 \notin L(ETOL)$ .

Proof.

If we set  $K_1 = W_1$ ,  $f$  equal to  $f_1$  restricted to  $W_1$  and  $K_2$  equal to the range of function  $f$  then we have  $W_2 = \{x f(x) : x \in K_1\}$  where  $f$  is a bijective function onto  $K_2$ . Thus if  $W_2 \in L(ETOL)$  then from Theorem 1 it follows that  $K_1$  is an EDTOL language which contradicts Lemma 1. Consequently  $W_2 \notin L(ETOL)$ .

Let  $f_2$  be a homomorphism from  $\{a, b\}^+$  onto  $\{0, 1\}^+$  defined by  $f_2(a) = 0$  and  $f_2(b) = 1$ . Let

$W_3 = \{x f_2(x) : x \in \{a, b\}^+ \text{ and } |x| = 2^n \text{ for some } n \geq 0\}$ .

Proposition 2.  $W_3 \notin L(ETOL)$ .

Proof.

If we set  $K_1 = \{x \in \{a, b\}^+ : |x| = 2^n \text{ for some } n \geq 0\}$ ,  $f$  equal to  $f_2$  restricted to  $K_1$  and  $K_2 = W_1$  then we have  $W_3 = \{x f(x) : x \in K_1\}$  where  $f$  is a bijective function onto  $K_2$ . Thus if  $W_3 \in L(ETOL)$  then from Theorem 1 it follows

that  $K_2$  is an EDTOL language which contradicts Lemma 1. Consequently  $W_3 \notin L(\text{ETOL})$ .

Let  $f_3$  be a homomorphism from  $\{a, b, c\}^+$  onto  $\{0, 1\}^+$  defined by  $f_3(a) = 0$ ,  $f_3(b) = 1$  and  $f_3(c) = \Lambda$ . Let  $f_c$  be a homomorphism from  $\{a, b, c\}^+$  onto  $\{0, 1\}^+$  defined by  $f_c(a) = a$ ,  $f_c(b) = b$  and  $f_c(c) = \Lambda$ . Let

$$W_4 = \{x f_3(x) : x \in \{a, b, c\}^+ \text{ and } |f_c(x)| = 2^n \text{ for some } n \geq 0\}.$$

Proposition 3.  $W_4 \notin L(\text{ETOL})$ .

Proof.

If we set  $K_1 = \{x \in \{a, b, c\}^+ : |f_c(x)| = 2^n \text{ for some } n \geq 0\}$ ,  $f$  equal to  $f_3$  restricted to  $K_1$  and  $K_2 = W_1$  then we have  $W_4 = \{x f(x) : x \in K_1\}$  where  $f$  is a function onto  $K_2$ . Thus if  $W_2 \in L(\text{ETOL})$  then from Theorem 2 it follows that  $K_2$  is an EDTOL language which contradicts Lemma 1. Consequently  $W_3 \in L(\text{ETOL})$ .

Note that the function  $f$  as defined in the proof of Proposition 2 is not a bijective function, hence it was necessary to apply Theorem 2 rather than Theorem 1.

Finally we can settle a quite important open problem of long standing (see, e.g., Downey [2] and Salomaa [16]) whether or not the class of ETOL languages is contained in the class of indexed languages (see Aho [1]). Let  $L(\text{IND})$  denote the class of indexed languages. (Now we assume that the reader is familiar with Aho [1]).

Theorem 3. Let  $\Sigma$  be a finite alphabet and let  $\bar{\Sigma} = \{\bar{a} : a \in \Sigma\}$ . Let  $h$  be a homomorphism from  $\Sigma^*$  onto  $\bar{\Sigma}^*$  defined by  $h(a) = \bar{a}$ , for every  $a$  in  $\Sigma$ . Let  $K$  be a context-free language over  $\Sigma$  such that  $K$  is not an EDTOL language. Then the language  $M_K^\dagger = \{w(h(w))^{\text{mir}} : w \in K\}$  is in  $L(\text{IND})$  but it is not in  $L(\text{EDTOL})$ .

Proof.

If a language is context-free then it can be generated by a right linear indexed right linear grammer (see Aho [1], Lemma 6.1). Thus, obviously,

$$M_K \in L(\text{IND}).$$

On the other hand from Theorem 1 it follows that  $M_K$  is not in  $L(\text{EDTOL})$ .

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<sup>†</sup>For a word  $x$ ,  $x^{\text{mir}}$  denotes the mirror image of  $x$ .

Now we turn to our next theorem.

Theorem 2. Let  $\Sigma_1, \Sigma_2$  be two disjoint alphabets and let  $K_1 \subseteq \Sigma_1^*, K_2 \subseteq \Sigma_2^*$ . Let  $f$  be a surjective function from  $K_1$  onto  $K_2$ . Let  $K = \{wf(w) : w \in K_1\}$ . If  $K$  is in  $L(ETOL)$  then

- (i)  $K_2 \in L(EDTOL)$ , and
- (ii) There exists  $\bar{K}_1$  such that  $\bar{K}_1 \subseteq K_1$ ,  $f(\bar{K}_1) = K_2$ ,  $\bar{K}_1$  is in  $L(EDTOL)$  and  $\{wf(w) : w \in \bar{K}_1\}$  is in  $L(EDTOL)$ .

Proof.

Most of the proof of this theorem was done already in the proof of Theorem 1. Let us note that in showing (in the proof of Theorem 1) that  $L(G) = L(G_1) = L(G_2) = L(G_3) = L(G_4)$  the particular property of the function  $f$  (its bijectiveness) was used only in proving that  $L(G_3) = L(G_4)$ .

Thus let  $G, G_1, G_2, G_3, G_4, G_4^{(1)}$  and  $G_4^{(2)}$  be defined as in the proof of Theorem 1.

As we still require that  $f$  is a surjective (but not necessarily bijective) function one can clearly see that  $L(G_4) = \{\gamma_1 \gamma_2 : \gamma_1 \in \Sigma_1^*, \text{ and } \gamma_2 \in \Sigma_2^*\}$  where for every  $\beta_1 \beta_2$  in  $K$ , where  $\beta_1 \in \Sigma_1^*$  and  $\beta_2 \in \Sigma_2^*$ , there exists  $\gamma_1$  in  $\Sigma_1^*$  such that  $\gamma_1 \beta_2$  is in  $L(G_4)$ . Also it is clear that  $L(G_4) \subseteq L(G_3)$  where  $L(G_3) = L(G) = K$ . Consequently

$$\{\gamma_1 \in \Sigma_1^* : \text{there exists } \gamma_2 \text{ in } \Sigma_2^* \text{ such that } \gamma_1 \gamma_2 \text{ is in } L(G_4)\} \subseteq K_1.$$

But then the theorem follows from the equalities:

$$K_2 = L(G_4^{(2)}), \bar{K}_1 = L(G_4^{(1)}) \text{ and } L(G_4) = \{wf(w) : w \in \bar{K}_1\}.$$

Hence Theorem 3 follows.

For each  $i \geq 1$ , let  $\Sigma_i = \{[\ ]_1, \dots, [\ ]_i, \dots, [\ ]_i, \dots, [\ ]_1\}$  and let  $B_i$  be the language generated by the context-free grammar  $H(B_i) = \langle \{S\}, \Sigma_i, P_i, S \rangle$ , where

$$P_i = \{S \xrightarrow{j} [SS]_j : 1 \leq j \leq i\} \cup \{S \xrightarrow{j} [S]_j : 1 \leq j \leq i\} \cup \{S \xrightarrow{j} [\ ]_j : 1 \leq j \leq i\}.$$

Let us recall now two results from Ehrenfeucht and Rozenberg [8]. (We assume the reader to be familiar with the notion of a Dyck language, see, e.g., Salomaa [4], p. 210).

Lemma 2. For every  $i \geq 1$ ,  $B_i$  is not in  $L(EDTOL)$ .

Lemma 3. If  $K$  is a Dyck language over an alphabet of at least eight letters then  $K$  is not in  $L(EDTOL)$ .

Now from Theorem 3, Lemma 2 and Lemma 3 we have the following results.

Corollary 1. For every  $i \geq 1$ ,  $M_{B_i} \in L(IND) - L(ETOL)$ .

Corollary 2. If  $K$  is a Dyck language over an alphabet of at least eight letters, then  $M_K \in L(IND) - L(ETOL)$ .



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