# A Note on Matrices of Trees *

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ABSTRACT

The notions of a matrix of trees and of a well formed matrix of trees are introduced. They arose from research in formal language theory. It is proved that each matrix of trees contains a "relatively large" submatrix which is well formed.

INTRODUCTION

One of the central notions of formal language theory is that of a derivation in a grammar (see e.g., Salomaa [2]). From a graph-theoretic point of view each derivation may be viewed as an arrangement of trees into a matrix. Using this approach we were able to discover a quite useful structure of derivations in the so-called deterministic ETOL systems (see Ehrenfeucht and Rozenberg [1]).

The notion of a <u>matrix of trees</u> which arose in this way is investigated in this paper. The main result presented here is used in a very essential way to prove the main result of Ehrenfeucht and Rozenberg [1].

DEFINITIONS AND EXAMPLES

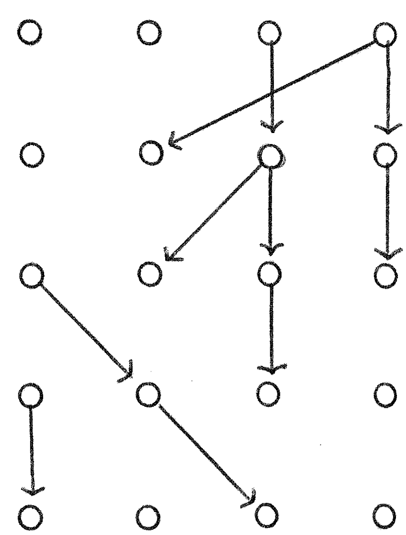
In this section we provide definitions (illustrated by examples) of the main notions used in this note.

Definition 1. A (n×k) matrix of trees (abbreviated as a (n×k) t-matrix) is a directed graph whose nodes form a (n×k) matrix which satisfies two conditions:

(i) each node in the graph has at most one ancestor,

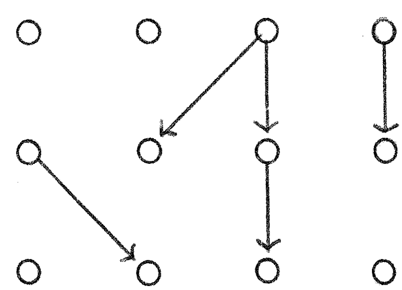
(ii) if there is an edge leading from node (i,j) to node $(\bar{i},\bar{j})$, for some $1 \leq i, \bar{i} \leq n$ and $1 \leq j, \bar{j} \leq k$, then $\bar{i} = i + 1$.

Example 1. The following is an example of a (5×4) t-matrix:



Definition 2. Let $G_1$ be a (n×k) t-matrix and let $G_2$ be a (m×k) t-matrix for some $m \leq n$. We say that $G_2$ is a sub-t-matrix of $G_1$ if the (m×k) matrix of nodes of $G_2$ is obtained by omitting some (maybe none) rows from the matrix of nodes of $G_1$ and there is an edge between two nodes in $G_2$ if and only if this edge is in the transitive closure of $G_1$.

Example 2. The (3×4) t-matrix shown below



is a sub-t-matrix of the t-matrix in Example 1. It is obtained by omitting the second and the fifth rows of the original matrix.

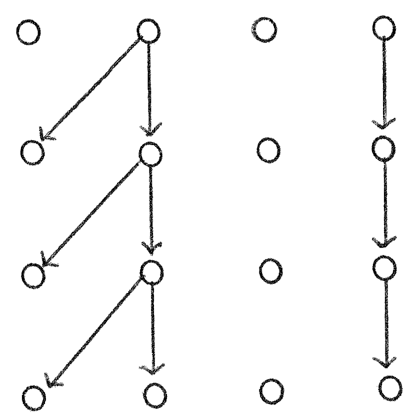
It should be clear to the reader that if $G_1$ is a sub-t-matrix of a t-matrix G and if $G_2$ is a sub-t-matrix of $G_1$, then $G_2$ is a sub-t-matrix of G.

Definition 3. A (n×k) t-matrix G is said to be well formed if it satisfies the following two conditions:

1) If a node (i,j) has descendants then (i+1,j) is one of them.

2) If there is an edge leading from (i,j) to (i+1,$\ell$), then, for every p in {1, ..., n-1}, there is the edge leading from (p,j) to (p+1,$\ell$).

Example 3. The t-matrices in examples 1 and 2 are not well formed. The following is an example of a well formed (5×4) matrix:



Given a t-matrix G we can in an obvious sense talk about its rows and columns.

Definition 4. Let G be a t-matrix and let $\alpha$ be one of its columns.

1) We say that $\alpha$ is a column of type 1 if from each node in $\alpha$ (except for the last one) there is an edge leading to the next node in $\alpha$.

2) We say that $\alpha$ is a column of type 2 if no node in $\alpha$ has a descendant and if for every node in $\alpha$ which has an ancestor, the ancestor belongs to a column of type 1.

3) We say that $\alpha$ is an arranged column if it is either of type 1 or of type 2.

Example 4. For the t-matrix of Example 3, the second and the fourth columns are of type 1 and the first and the third columns are of type 2.

It should be clear to the reader that if a column is arranged in a t-matrix G, then it "stays arranged" in all sub-t-matrices of G.

MAIN RESULT

In this section we shall prove that each t-matrix has a "relatively large" well formed sub-t-matrix.

Lemma 1. Let G be a $(n \times k)$ t-matrix which has $\ell$ arranged columns for some $1 \leq \ell \leq k-1$. There exists a sub-t-matrix $G_1$ of G such that

1) $G_1$ has at least $(\ell+1)$ arranged columns, and

2) $G_1$ is of order $(n_1 \times k)$ for some $n_1 \geq \frac{\sqrt{n}}{k}$.

Proof.

Let G satisfy the statement of the lemma.

Let $P_G$ be the collection of all paths in G the nodes of which do not belong to any of the arranged columns in G. Let $\gamma$ be a path from $P_G$ such that no path in $P_G$ is longer than $\gamma$.

(i) Let us assume that the length of $\gamma$ is larger than $\sqrt{n}$.

Let $C_\gamma$ be the set of columns which have at least one node that belongs to $\gamma$. Let $c_\gamma$ be a column from $C_\gamma$ such that no other column in $C_\gamma$ has more nodes in $\gamma$ than $c_\gamma$ has. Clearly $c_\gamma$ has at least $\frac{\sqrt{n}}{k}$ nodes in $\gamma$. Thus if we choose $G_1$ to be the sub-t-matrix of G consisting of all the rows of G having a node of $\gamma$ in their $c_\gamma$ column, then we see that $G_1$ satisfies conditions 1) and 2) and Lemma 1 holds.

(ii) Let us assume that the length of $\gamma$ is no larger than $\sqrt{n}$.

Now if we choose all the rows numbered† $1, \lceil\sqrt{n}\rceil, \lceil\sqrt{n}\rceil \cdot 2, \ldots$ we obtain a sub-t-matrix $G_1$ satisfying the statement of the Lemma.

This ends the proof of Lemma 1.

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†For a real number r, $\lceil r \rceil$ denotes the smallest integer larger than r.

Lemma 2. Let G be a $(n\times k)$ t-matrix. Then there exists a sub-t-matrix H of G such that

1) all columns of H are arranged, and

2) H is of order $(m\times k)$ for some $m \geq \frac{n^{\frac{1}{2k}}}{k^k}$.

Proof

The result follows immediately from repeated (at most k times) application of lemma 1.

Now we can easily prove our main result, which, informally speaking, says that each t-matrix has a "relatively large" well formed sub-t-matrix.

Theorem. For every positive integer k there exist positive reals $r_k$ and $s_k$ such that for every positive integer n and for every $(n\times k)$ t-matrix G there exists a well formed $(m\times k)$ t-matrix H which is a sub-t-matrix of G and for which $m \geq r_k n^{s_k}$.

Proof.

Let k be a positive integer and let G be a $(n\times k)$ t-matrix. By Lemma 2, there exists a sub-t-matrix $\overline{H}$ of G of order $(p\times k)$, for some $p \geq \frac{n^{\frac{1}{2k}}}{k^k}$, in which each column is arranged.

Let $i,j\varepsilon\{1, \ldots, p-1\}$. We say that the $i^{th}$ and the $j^{th}$ rows of $\overline{H}$ are isomorphic if the following holds: for every $t_1$, $t_2$ in $\{1, \ldots, k\}$, there is an edge leading from $(i,t_1)$ to $(i+1,t_2)$ if and only if there is an edge leading from $(j,t_1)$ to $(j+1,t_2)$. By a type in $\overline{H}$ we mean a set of all of which are isomorphic rows of $\overline{H}$. Let T be a type in $\overline{H}$ such that no type in $\overline{H}$ contains more elements than T. As the number of different types is certainly no larger than $2^{k^2}$, the number of elements in T is at least $\frac{n^{\frac{1}{2k}}}{k^k} \cdot \frac{1}{2^{k^2}}$. If we choose now H to be the sub-t-matrix of $\overline{H}$ consisting of all these rows of $\overline{H}$ which are in T then it is clear that H is well formed. Consequently if we choose $r_k = \frac{1}{k^k 2^{k^2}}$ and $s_k = \frac{1}{2k}$ then H satisfies the statement of the theorem.

Thus the theorem holds.

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