Principal Minor Determinant Formulas*

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1. Introduction. Our purpose in these notes is to provide a reasonably complete survey of the known formulas for expanding determinants of matrices in terms of principal minors and to explain these formulas in graph theoretic terms. Some of the results presented have been published (see [1], [2]) or used in already published papers (see, for example, [3], [4], [5], [6], [7]) but nowhere has a systematic account of these results been given. We shall also present several new formulas which have not yet been published.

We shall denote by $\mathcal{P}_n$ the set of the first n integers $\mathcal{P}_n = (1\ 2\ \ldots\ n)$. By a path in $\mathcal{P}_n$ we mean an ordered sequence $(i_1\ i_2\ \ldots\ i_s)$, $s \geq 2$, of integers where each $i_j \in \mathcal{P}_n$. We say that the path has length $s-1$. A path will be called a chain if the integers $i_1, \ldots, i_s$ are distinct. Thus $s \leq n$ for a chain. If all of the indices in a path are distinct except for the first and last indices we call the path a cycle ($s \leq n+1$ for a cycle). These concepts relative to the set $\mathcal{P}_n$ can be extended readily to matrices.

Definition 1. Let $A = [a_{ij}]_1^n$ be a square matrix over any field $\mathcal{F}$.

i) By a chain in A we mean any product

$$a(i_1 \ldots i_s) = \prod_{k=1}^{s-1} a_{i_k i_{k+1}}$$

where $(i_1 \ldots i_s)$ is a chain in $\mathcal{P}_n$.

ii) By a cycle in A we mean any product

$$\hat{a}(i_1 \ldots i_s) = \left(\prod_{k=1}^{s-1} a_{i_k i_{k+1}} \right) a_{i_s i_1}$$

where $(i_1 \ldots i_s i_1)$ is a cycle in $\mathcal{P}_n$, $s \geq 1$.

If $(i_1 \ldots i_s)$ is a chain in $\mathcal{P}_n$ we call i_1 the initial index of the chain and i_s the terminal index of the chain. Thus if j and k are fixed indices in $\mathcal{P}_n$ and we wish to indicate a chain in A with initial index j and terminal index k it is natural to use the notation $a(j \to k)$.

We require also the following concepts. Let $H = (h_1,\ldots,h_r)$ be a subset of $\mathcal{P}_n$ such that the indices satisfy $h_1 < h_2 < \ldots < h_r$. We call H an increasing multi-index of length r . The set of all increasing multi-indices of length r will be denoted by $\alpha(n,r)$. The set $\alpha(n,r)$ contains $\binom{n}{r}$ elements. If $H \in \alpha(n,r)$ we denote by H' the compliment of H in $\mathcal{P}_n$ regarded as an element of $\alpha(n,n-r)$.

If $H \in \alpha(n,p)$ and $K \in \alpha(n,p)$ we shall let $A(H,K)$ denote the submatrix of A contained in the rows H and columns K . Thus

$$A(H,K) = [a_{h_i k_j}]_1^p .$$

When $H = K$ we write $A(H)$ in place of $A(H,H)$. The submatrices $A(H)$ are the principal submatrices of A . When $H = \mathcal{P}_n$ we have $A(\mathcal{P}_n) = A$ hence A is a principal submatrix of itself.

<u>Definition 2</u>. Let $H \in \alpha(n,p)$. We shall denote by $A_{(H)}$ the sum of all of the cycles of length p (p-cycles) in $A(H)$.

We remark that there are $(p-1)!$ p-cycles in $A(H)$ where $H \in \alpha(n,p)$.

As an example to illustrate these concepts suppose $n = 5$ so $A = [a_{ij}]_1^5$ and $H = (145)$. Then

2. The graph of the matrix A . It is by now widely understood that many results in matrix theory can be explained most conveniently with the help of graphs. We shall indeed rely heavily upon this tool in these notes.

Definition 3. Let $A = [a_{ij}]_1^n$ be a square matrix. The digraph (directed graph) of A has vertex set $\mathcal{P}_n$ and a (directed) edge (i,j) whenever $a_{ij} \neq 0$. We denote the digraph of A by D(A) .

All concepts relative to a digraph such as path, chain, or cycle should be mentioned with the adjective directed, but we shall usually ignore this and speak simply of chains and cycles. We remind the reader that a path in D(A) is an ordered sequence of integers $(i_1 \ldots i_s)$, $s \geq 2$, such that each edge (i_j, i_{j+1}) , $1 \leq j \leq s-1$, belongs to D(A) . We then define chains and cycles in D(A) in the same way as was done in the introduction for $\mathcal{P}_n$.

Observe that our point of view here is different from that used in the introduction. There we placed no restrictions upon the possible paths, chains, and cycles. From the graph theoretic point of view $\mathcal{P}_n$ was regarded as being the vertex set of a complete digraph so that for any i and j , the edge (i,j) could be assumed to be present. This enabled us to define chains and cycles in the matrix A in a completely general way. Now, by associating D(A) with A , we can concentrate only upon those chains and cycles of A which are not zero. Indeed, it is clear that there is a 1-1 correspondence between the nonzero cycles of A and the cycles of D(A) and a 1-1 correspondence between the nonzero chains of A and the chains of D(A) .

Another observation is in order. Our digraph D(A) has loops. Many authors do not allow loops in digraphs but we find them essential for our purposes. The loops are the 1-cycles of D(A) and correspond to the elements on the principal diagonal of A .

Now obviously one reason for introducing D(A) is to enable us to give pictorial representations of the structure of the matrix A . As an example consider any matrix of the form

$$A = \begin{bmatrix} 0 & x & x & 0 & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & x & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x & x & 0 \end{bmatrix}$$

where the x's denote the locations of nonzero elements. Then we have

D(A):

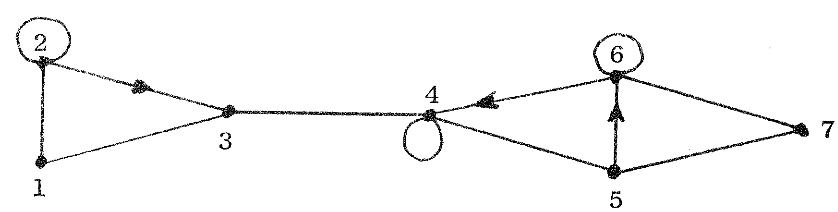


Figure 1

Here and elsewhere we follow the convention of using an undirected edge between two vertices whenever both directed edges are present. This should cause no confusion. Also it is not necessary to put an arrowhead on any of the loops.

Among the nonzero cycles of any such matrix A as shown above are $a_{45}a_{57}a_{76}a_{64}$, which is indeed the only nonzero 4-cycle of A , and $a_{12}a_{23}a_{31}$. Each pair of distinct vertices is connected by a chain, for example, $a_{21}a_{13}a_{34}a_{45}a_{57}a_{76}$ is a chain of length 6 .

3. The first determinant formula. We begin our discussion of principal minor determinant formulas with

Theorem 1. Let $H \in \alpha(n, n-1)$ be fixed. Then

$$d(A) = a_{H'H'} A_H + \sum_{r=0}^{n-2} (-1)^{n+1-r} \sum_{K \in \alpha(H,r)} A_K A_{(K')} . \tag{1}$$

Before proving Theorem 1 some remarks are in order. Note first that H' is a single index, hence $a_{H'H'}$ is simply a 1-cycle of A. Moreover, we have used the symbol $\alpha(H,r)$ to denote the set of multi-indices of length r, $0 \le r \le n-2$, contained in the set H. When $r = 0$ this set is empty and we set $A_K = 1$. This convention will be used henceforth.

Proof of Theorem 1. For definiteness we shall suppose that $H' = n$. Recall that

$$\det A = \sum_{\sigma(\mathcal{P}_n)} \operatorname{sgn} \sigma \, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} .$$

Consider the set of permutations of $\mathcal{P}_n$ for which $\sigma(n) = n$. There is a 1-1 correspondence between this set and the set of permutations of $\mathcal{P}_{n-1}$. Thus we can write

$$\det A = a_{nn} \sum_{\sigma(\mathcal{P}_{n-1})} \operatorname{sgn} \sigma \, a_{1,\sigma(1)} \cdots a_{n-1,\sigma(n-1)}$$

$$+ \sum_{\sigma(\mathcal{P}_n)}{}' \operatorname{sgn} \sigma \, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$$= a_{nn} A_H + \sum_{\sigma(\mathcal{P}_n)}{}' \operatorname{sgn} \sigma \, a_{1\sigma(1)} \cdots a_{n\sigma(n)} , \tag{2}$$

where the prime on the summation means the terms corresponding to permutations of $\mathcal{P}_n$ leaving n and $\mathcal{P}_{n-1}$ invariant are omitted. To complete the proof consider next the set of permutations of $\mathcal{P}_n$ such that $a_{j_1\sigma(j_1)}a_{j_2\sigma(j_2)} \cdots a_{j_{r-1}\sigma(j_{r-1})}a_{j_r\sigma(j_r)}$ is a fixed cycle of length $r > 1$ in A. Let K be the set of indices complimentary to $(j_1,\ldots,j_n)$ in $\alpha(n,n-r)$. Our set of permutations is in 1 - 1 correspondence to the set of permutations $\sigma(K)$. Note that the set $(j_1 \ldots j_r)$ must contain n. Moreover, we have

$$\operatorname{sgn} \sigma(\mathcal{P}_n) = (-1)^{n+1-r} \operatorname{sgn} \sigma(K)$$

for any such permutation of $\mathcal{P}_n$. Thus the contribution of this set to $\sum'$ in (2) is

$$(-1)^{n+1-r} a_{j_1\sigma(j_1)} \cdots a_{j_{r-1}\sigma(j_{r-1})} a_{j_r\sigma(j_r)} A_K$$

where $a_{j_1\sigma(j_1)} \cdots a_{j_r\sigma(j_r)}$ is a fixed r - cycle in $A(K')$. It only remains for us to establish that all of the sum $\sum'$ in (2) is exhausted by such terms. But this follows from the fact that every permutation $\sigma(\mathcal{P}_n)$ can be written uniquely as the product of cycles and all of those in which the 1 - cycle a_{nn} occurs appear in the first term of (2). Thus theorem 1 is proved.

There are some important facts that should be noted before we turn to some examples illustrating the formula. Suppose that n - r is even so that each summand in $A_{(K')}$ is a cycle of even length. Then the corresponding term in formula (1) has a negative sign attached. On the other hand, if n - r is odd the corresponding term has a positive sign attached. Hence we have the following rule of signs for the terms in formula (1): Each term corresponding to a cycle of even length has a negative sign attached and each term corresponding to a cycle of odd length has a positive sign attached.

In view of the above remarks we can rewrite formula (1) in the following way. Let i be a fixed index and set $H = i' = (1\ 2 \ldots i-1\ i+1 \ldots n)$. Then

$$d(A) = a_{ii} A_H + \sum_{r=2}^{n} (-1)^{r-1} \sum_{k \in \alpha(H,n-r)} A_K A_{(K')} \ . \tag{1'}$$

Next we point out that the formula may be thought of as an expansion formula relative to a fixed principal diagonal element of the matrix A. Each cycle which appears explicitly in (1) contains the index H'. This fact leads us at once into a graph theoretic interpretation of the formula which is embodied in the following proposition.

Proposition 1. The only possible nonzero terms in the formula (1) are those which correspond to cycles of $D(A)$ which contain the vertex H'.

With the above facts at our disposal let us turn to some examples illustrating the theorem.

Example 1. Consider any matrix having the form given in section 2. From the digraph illustrated in figure 1 we see that the vertex 1, for example, is contained in exactly 3 nonzero cycles, namely

$$a_{12}a_{21}\ , \quad a_{13}a_{31}\ , \quad \text{and} \quad a_{12}a_{23}a_{31}\ .$$

Therefore by formula (1) we must have

$$d(A) = -a_{12}a_{21} A_{34567} - a_{13}a_{31} A_{24567} + a_{12}a_{23}a_{31} A_{4567}\ , \tag{3}$$

for any such matrix. The formula therefore reduces the computation of the seventh order determinant to that of two fifth order determinants and one fourth order determinant.

In example 1 we deliberately choose to expand relative to an index contained in a minimal number of cycles resulting in as few nonzero terms as possible in the formula. That this is not always the best policy is illustrated by the next example.

Example 2. Let $A = [a_{ij}]_1^n$ with $a_{ii} = a_i$, $1 \le i \le n$, $a_{i,i+1} = b_i$, $1 \le i \le n-1$, $a_{i+1,1} = c_i$, $1 \le i \le n-1$, and $a_{ij} = 0$ otherwise. The graph of A is shown in figure 2. The number of cycles incident

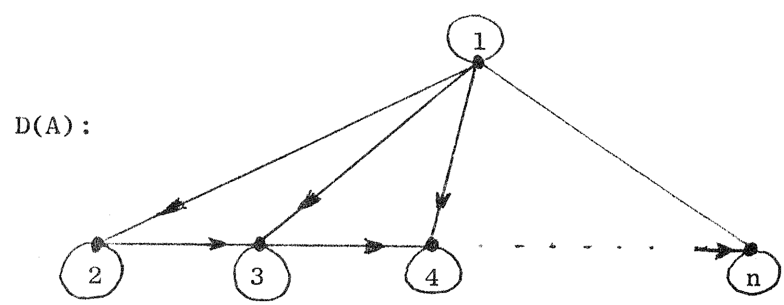


Figure 2

at vertices one and two is n and the number incident at vertex k for $k > 2$ is $n-k+2$. Nevertheless it is somewhat simpler to use formula (1) on vertex 1 because the corresponding principal submatrices are all upper triangular. The 1-cycle at 1 is a_1 and the corresponding principal minor is $\prod_{j=2}^{n} a_j$. The k-cycle at 1 is $b_1 b_2 \ldots b_{k-1} c_{k-1}$ and the corresponding principal minor is $\prod_{j=k+1}^{n} a_j$.

Thus we arrive at

$$d(A) = \prod_{i=1}^{n} a_i + \sum_{k=2}^{n} (-1)^{k+1} c_{k-1} \prod_{j=1}^{k-1} b_j \prod_{j=k+1}^{n} a_j .$$

It is of interest to observe here that

$$d(A-\lambda I) = \prod_{i=1}^{n} (a_i - \lambda) + \sum_{k=2}^{n} (-1)^{k+1} c_{k-1} \prod_{j=1}^{k-1} b_j \prod_{j=k+1}^{n} (a_j - \lambda) .$$

Let us observe next that formula (1) applies equally well to all principal minors of the matrix A. Associated with each principal submatrix $A(H)$ of A is the digraph $\langle H \rangle$ which is the vertex induced subgraph of $D(A)$ with vertices $(h_1 \dots h_p)$ (if $H \in \alpha(n,p)$).

To illustrate, consider again example 1. According to formula (3) the principal minors A_{34567}, A_{24567}, and A_{4567}, occur in the expansion of $d(A)$. The corresponding subgraphs are as shown in figure 3, (a), (b), (c) respectively.

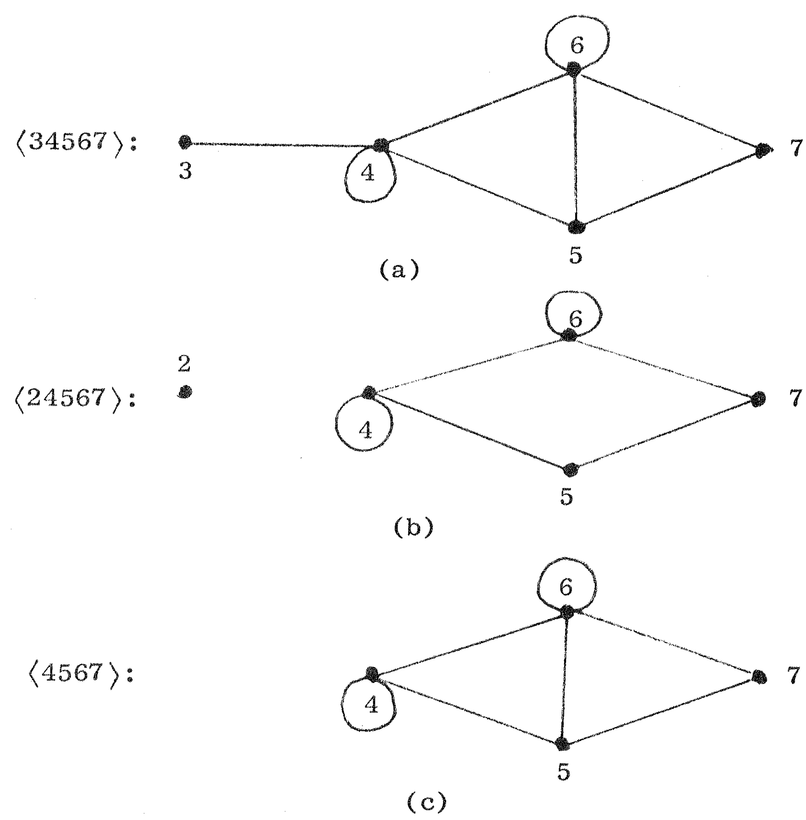


Figure 3

Observe, for example that $\langle 24567 \rangle$ is not connected hence A_{24567} is the direct sum of the 1×1 matrix a_{22} and the 4×4 matrix A_{4567}. Thus $A_{24567} = a_{22} A_{4567}$. This result could be substituted into formula (3) to simplify it somewhat.

4. Cycle bases and expansion with respect to a fixed principal submatrix. Let i be a fixed vertex in $D(A)$. By the cycle bases at i we simply mean the set of cycles of $D(A)$ which contain the vertex i. Similarly we can define the cycle bases at any subset of vertices of $D(A)$ as follows.

Two cycles of $D(A)$ will be called disjoint if they contain no common vertices. A set C of cycles in $D(A)$ spans the vertex set X if every cycle in C contains at least one vertex from the set X, every vertex in X appears in a cycle in C, and the cycles are pairwise disjoint. The cycle basis for the set X is the set of all spanning sets of cycles for X.

As an example consider the graph of figure 1 and take X to be the set $X = \{1,2\}$. The cycle basis at X is then

$$\{\{(121)\}, \{(1231)\}, \{(22),(131)\}\} .$$

Now if we have the cycle basis at i in $D(A)$ this corresponds to the cycle basis at the index i in A. Similarly to the cycle basis at the subset $H = (h_1 \ldots h_p)$, $H \in \alpha(n,p)$, there corresponds the cycle basis in A. For the above example this would be $\{\{a_{12}a_{21}\}, \{a_{12}a_{23}a_{31}\}, \{a_{22}, a_{13}a_{31}\}\}$. Let $\hat{a}$ be a cycle in the cycle basis at i, then the principal minor of A defined by the indices complimentary to those appearing in $\hat{a}$ will be called the cominor of $\hat{a}$. Similarly if we have an element of the cycle basis at H, $H \in \alpha(n,p)$, the principal minor defined by the indices complimentary to those appearing in the cycles is called the cominor of the spanning set.

We can use these concepts and theorem 1 to obtain an interesting and useful extension of the formula (1). For this purpose let $H \in \alpha(n,p)$ be fixed and let $B_H = \{c_1, \ldots, c_m\}$ be the cycle basis in A at H. With the element $c_j \in B_H$ we associate the numbers p_j, the product of the cycles in c_j, and $(-1)^{\nu_j}$ where ν_j is the number of cycles of even length in c_j. Let K_j be the set of indices from $\mathcal{P}_n$ not appearing in c_j so that A_{K_j} is the cominor of c_j. Then the following formula results from repeated application of theorem 1:

(Fundamental Principal Minor Formula)

$$d(A) = \sum_{j=1}^{m} (-1)^{\nu_j} p_j A_{K_j} \tag{3}$$

We regard formula (3) as being the fundamental principal minor determinant formula. From it all other formulas can be derived.

Example 3: Formula (1) is the special case of formula (3) when $H \in \alpha(n,1)$ and B_H is the cycle basis in A at a single index.

Example 4. Let $H = \mathcal{P}_n$ then each element of $B_H = B$ is called a linear spanning set for $D(A)$. Each cominor becomes equal to 1 and the formula (3) becomes

$$d(A) = \sum_{j=1}^{m} (-1)^{\nu_j} p_j . \tag{4}$$

This formula appears to be due to Goldberg [2], but it first was put into the above form by Harary [8].

Example 5. Consider once again any matrix whose digraph is given in figure 1. Choose $H = (123)$. We have

$B_H = \{\{a_{12}a_{23}a_{31}\}, \{a_{13}a_{31}, a_{22}\}, \{a_{12}a_{21}, a_{34}a_{43}\}\}$. The corresponding cominors are A_{4567}, A_{4567}, A_{567} respectively. The formula (3) therefore gives

$$d(A) = a_{12}a_{23}a_{31}\, A_{4567} - a_{22}a_{13}a_{31}\, A_{4567} + a_{12}a_{21}a_{34}a_{43}\, A_{567} .$$

We can refine the formula (3) with the help of example 4. In the cycle basis B_H we can distinguish two types of elements. A spanning set of cycles in A for H will be called minimal if the set of indices appearing in the set contains no elements of H' . For each minimal spanning set of cycles for H the cominor is $A_{H'}$. Clearly, if we sum over the terms in formula (3) which correspond to minimal spanning sets we obtain simply $A_H A_{H'}$. Let us denote by E_H the subset of spanning sets of cycles of H which are not minimal. If $C \in E_H$ let $p(c)$ be the product of the cycles in the spanning set c and let $\nu(c)$ be the number of even cycles in C . Then (3) becomes

$$d(A) = A_H A_{H'} + \sum_{C \in E_H} (-1)^{\nu(c)} p(c) A_{K(c)} \qquad (5)$$

where $A_{K(c)}$ is the cominor of c . The set E_H will be called the set of essential spanning sets of H .

Observe that we can expand $d(A)$ relative to the principal minor A_H even if $A_H = 0$. Observe also that by formula (5) the matrix A is indecomposable iff there exists an $H \in \alpha(n,p)$ for some $1 \le p \le n$, such that $d(A) = A_H A_{H'}$. Finally observe that each cominor $A_{K(c)}$ for $c \in E_H$ is a principal minor of $A(H')$ of order $< n-p$.

In the form (5) formula (3) appears as a direct generalization of formula (1). It plays the same role in the theory of principal minor determinant formulas as the general Laplace expansion formula does in the classical determinant formulas.

Now we can return to the point of view of section 1 and derive a formula similar to the formula (1). As in formula (5) let H and H' be fixed, $H \in \alpha(n,p)$. For $r \geq 2$, $1 \leq q \leq \min(p,r-1)$, let $\alpha((H,H'),(q,r-q))$ denote the set of multi-indices of length r with q indices in H and $r-q$ indices in H'. If $K \in \alpha((H,H'),(q,r-q))$ we shall denote by $A(H-K)$ the principal submatrix of $A(H)$ obtained by deleting the q indices in $H \cap K$. $A(H'-K)$ is similarly defined. Then formula (5) becomes

$$d(A) = A_H A_{H'} + \sum_{r=2}^{n} (-1)^{r+1} \sum_{q=1}^{p} \sum_{k \in \alpha((H,H'),(q,r-q))} A_{(K)} A_{H-K} A_{H'-K} . \qquad (5')$$

As an illustration of the formula (5′) consider for $n = 5$, $H = (13)$, $H' = (245)$. Then

$$A_H A_{H'} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} a_{22} & a_{24} & a_{25} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix} .$$

There will be six terms for $r = 2$, $p = 1$ of which a typical term is

$$-a_{14}a_{41}a_{33} \begin{vmatrix} a_{22} & a_{25} \\ a_{52} & a_{55} \end{vmatrix} .$$

There will be six terms for $r = 3$, $p = 1$, a typical term being

$$(a_{12}a_{25}a_{51} + a_{15}a_{52}a_{21})a_{33}a_{44} .$$

There will be three terms for $r = 3$, $p = 2$, a typical term being

$$(a_{13}a_{32}a_{21} + a_{12}a_{23}a_{31}) \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} .$$

There will be three terms for $r = 4$, $p = 2$, two terms for $r = 4$, $p = 1$, and one term for $r = 5$.

5. The cofactor formula and the quasi-principal minor formula.

Let σ, τ, $\sigma \neq \tau$, be fixed indices in $\mathcal{P}_n$ and let $\bar{A}_{\sigma\tau}$ be the complimentary minor in A to the element $a_{\sigma\tau}$, i.e., $\bar{A}_{\sigma\tau}$ is the determinant of the submatrix $A(\sigma', \tau')$ of A obtained by deleting row σ and column τ. After $\sigma + \tau - 3$ interchanges of rows and columns this determinant can be put into the form

$$\bar{A}_{\sigma\tau} = (-1)^{\sigma+ \tau-3} \begin{vmatrix} a_{\tau\sigma} & a_{\tau 1} \cdots a_{\tau n} \\ a_{1\sigma} & \\ \vdots & A((\sigma\tau)') \\ a_{n\sigma} & \end{vmatrix},$$

where, as the notation implies, $A((\sigma\tau)')$ is the principal submatrix of A of order $n-2$ consisting of the elements in all rows and columns except σ and τ.

If we now apply formula (1) relative to the element $a_{\tau\sigma}$ we arrive immediately at the formula

$$\bar{A}_{\sigma\tau} = (-1)^{\sigma+\tau-3}\{a_{\tau\sigma} A_{(\sigma\tau)'} + \sum_{r=0}^{n-3} (-1)^{n-r} \sum_{H \in \alpha((\sigma\tau)', r)} A_H A(\sigma', \tau')_{(H')}\} . \qquad (6)$$

Here the only notational difficulty would be with the term $A(\sigma', \tau')_{(H')}$ which is the sum over all $n-1-r$ cycles in the submatrix $A(\sigma', \tau')$ complimentary to $A(H)$. From the combinatorial point of view, however, the formula requires more explanation which we defer until after we write out the cofactor formula.

Let us denote by $A_{\sigma\tau}$ the algebraic cofactor of the element $a_{\sigma\tau}$, $\sigma \neq \tau$, of A so that $A_{\sigma\tau} = (-1)^{\sigma+\tau} \bar{A}_{\sigma\tau}$. Then we have

$$A_{\sigma\tau} = -[a_{\tau\sigma} A_{(\sigma\tau)'} + \sum_{r=0}^{n-3} (-1)^{n-r} \sum_{H \in \alpha((\sigma\tau)', r)} A_H A(\sigma', \tau')_{(H')}] . \qquad (7)$$

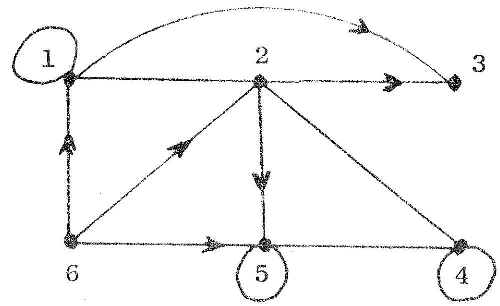
Let us begin by relating the graph of $A(\sigma',\tau')$ to the graph of A itself. The nonzero elements in row σ of A correspond to edges in D(A) with σ as initial vertex and the nonzero elements in column τ of A correspond to edges in D(A) with τ as terminal vertex. Consequently $D(A(\sigma',\tau'))$ is obtained from D(A) by deleting all such edges. It is the subgraph of D(A) in which the vertex τ is a source and the vertex σ is a sink.

Now we shall perform the following construction. Let $D_0(A(\sigma',\tau'))$ be the digraph obtained from $D(A(\sigma',\tau'))$ by replacing the pair of vertices σ and τ by a single vertex denoted by $\langle\tau,\sigma\rangle$. Thus the vertices of $D_0(A(\sigma',\tau'))$ are $(\mathcal{P}_n - \{\sigma,\tau\}) \cup \{\langle\tau,\sigma\rangle\}$. Since τ is a source and σ a sink in $D(A(\sigma',\tau'))$, $D_0(A(\sigma',\tau'))$ is a digraph. The cycle basis in $D_0(A(\sigma',\tau'))$ at the vertex $\langle\tau,\sigma\rangle$ clearly consists of the chains in $D(A(\sigma',\tau'))$ from τ to σ. Thus the nonzero terms in $A(\sigma',\tau')_{(H')}$ in formulas (6) or (7) correspond to the nonzero chains in A from τ to σ. These chains play the same role in the cofactor formula that the cycles play in formula (1).

Example 6. Consider any matrix having the form

$$A = \begin{bmatrix} x & x & x & 0 & 0 & x \\ x & 0 & x & x & x & 0 \\ 0 & 0 & x & x & 0 & 0 \\ 0 & x & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & x \\ x & x & 0 & 0 & x & 0 \end{bmatrix} .$$

The cofactor graph $D(A(3',6'))$ is as shown in figure 4(a), and $D_0(A(3',6'))$ is shown in figure 4(b).



(a)

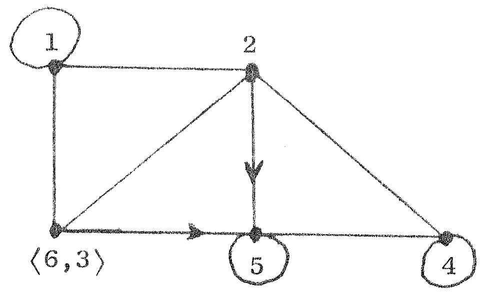


Figure 4(b)

The cycle basis at $\langle 6,3\rangle$ is

$$\{(\langle 6,3\rangle,1,2,\langle 6,3\rangle)\ ,\ (\langle 6,3\rangle,2,\langle 6,3\rangle)\ ,\ (\langle 6,3\rangle,5,4,2,\langle 6,3\rangle)\ ,\ (\langle 6,3\rangle,5,4,2,1,\langle 6,3\rangle$$

with corresponding chains in A given by

$$\{a_{61}a_{12}a_{23}\ ,\ a_{62}a_{23}\ ,\ a_{65}a_{54}a_{42}a_{23}\ ,\ a_{65}a_{54}a_{42}a_{21}a_{13}\}$$

Therefore, according to formula (7)

$$A_{36} = -a_{61}a_{12}a_{23}A_{45} + a_{62}a_{23}A_{145} + a_{65}a_{54}a_{42}a_{23}a_{11} - a_{65}a_{54}a_{42}a_{21}a_{13}\ .$$

Observe that the rule of signs for terms in either of the formulas (6) or (7) is the following: Each term corresponding to a chain of even length has a positive sign attached and each term corresponding to a chain of odd length has a negative sign attached.

The cofactors of A are important because of their close connection with the inverse of A, but they are a special case of the following more general class of minors of the matrix A.

<u>Definition 3</u>. The minor $A_{H,K}$ is called a quasi-principal minor of the matrix A if $H \in \alpha(n,p)$, $K \in \alpha(n,p)$ and $H \cup K \in \alpha(n,p+1)$.

(Note: by $H \cup K \in \alpha(n,p+1)$ we mean that the corresponding ordered set is in $\alpha(n,p+1)$.)

We can apply the formula (1) (or (5)) to a quasi-principal minor in the following way. Let $L \in \alpha(n,p+1)$ contain the distinct indices in H and K , i.e., L is the properly ordered set $H \cup K$. Let $L - H = h$, $L - K = k$. Then $A(H,K)$ is the submatrix of $A(L)$ obtained by deleting row h and column k from $A(L)$. Thus $A_{H,K}$ is the complimentary minor to the element a_{hk} in the submatrix $A(L)$. Let $\mu(h)$ be the number of elements in K which precede h and $\mu(k)$ be the number of elements in H which precede k. We may therefore apply the formula (6) to obtain

$$A_{H,K} = (-1)^{\mu(h)+\mu(k)-3}\Big[a_{kh}A_{H\cap K} + \sum_{r=0}^{p-3} (-1)^{p-r} \sum_{I \in \alpha(H \cap K, r)} A_I A(H,K)_{(I')}\Big] . \qquad (8)$$

Formula (8) is the general formula for the expansion of a quasi-principal minor of A in terms of principal minors of A .

Let us analize this formula from the graph theoretic point of view. Consider the vertex induced subgraph $\langle H \cup K\rangle$. Let $D\langle H \cup K\rangle$ denote the graph obtained from $\langle H \cup K\rangle$ by deleting edges so that vertex k becomes a source and vertex h becomes a sink. Finally let $D_0\langle H \cup K\rangle$ be obtained from $D\langle H \cup K\rangle$ by joining the pair of vertices h and k into the ordered vertex pair $\langle k,h\rangle$. Then the nonzero terms in formula (8) are those corresponding to the cycle basis in $D_0\langle H \cup K\rangle$ at the vertex pair $\langle k,h\rangle$. This is precisely the set of nonzero chains in the submatrix $A(H \cup K)$ from k to h .

Example 7. Among the quasi-principal minors of A an important subset consists of the almost-principal minors first introduced by Gantmacher and Krein [9]. The minor $A_{H,K}$ is called almost principal if among the differences $h_i - k_i$, $1 \le i \le p$ $(H \in \alpha(n,p)$, $K \in \alpha(n,p))$, exactly one is different from zero. For such minors $\mu(h) = \mu(k)$ and we have

$$A_{H,K} = -[a_{kh}A_{H \cap K} + \sum_{r=0}^{p-3} (-1)^{p-r} \sum_{I \in \alpha(H \cap K, r)} A_I A(H,K)_{(I')}] . \qquad (8')$$

In order to clarify the above concepts we note that $A_{12347,23578}$ is a quasi-principal minor but not an almost principal minor. On the other hand, $A_{12357,12457}$ is an almost principal minor.

6. The general formula for minor determinants. We turn next to a formula which furnishes the expansion of an arbitrary minor of A in terms of principal minors of A. The development combines all of the previous ideas and we shall start from the graph of A. Suppose, in fact, that $H \in \alpha(n,p)$, $K \in \alpha(n,p)$, $H \neq K$. Let us again set $L = H \cup K$, $L \in \alpha(n,q)$ for some $p < q \leq n$. $L - H = (h_1, \ldots, h_s)$, $L - K = (k_1, \ldots, k_s)$ where $s = q - p$ and $h_1 < h_2 < \ldots < h_s$, $k_1 < k_2 < \ldots < k_s$. Define the numbers $\mu(h_j)$, $\mu(k_j)$, $1 \leq j \leq s$, as the number of elements of $H \cap K$ which precede h, respectively, the number of elements of $H \cap K$ which precede k.

Now consider the subgraph $\langle L \rangle$ of $D(A)$. Let $D\langle L \rangle$ denote the graph obtained from $\langle L \rangle$ by deleting edges so that each vertex $k_1, \ldots, k_s$ becomes a source and each vertex $h_1, \ldots, h_s$ becomes a sink. Then let $D_0\langle L \rangle$ be obtained from $D\langle L \rangle$ by joining the pairs h_j and k_j, $1 \leq j \leq s$, into the ordered vertex pairs $\langle k_j, h_j \rangle$. The graph $D_0\langle L \rangle$ has p vertices consisting of the s ordered vertex pairs $\langle k_j, h_j \rangle$, $1 \leq j \leq s$, and $p - s$ ordinary vertices. It is now only necessary to apply formula (3) using the cycle basis at the set $(\langle k_1, h_1 \rangle, \ldots, \langle k_s, h_s \rangle) = L^*$. Clearly this cycle basis consists of products of chains having initial vertices in the set $(k_1, \ldots, k_s)$ and terminal vertices in the set $(h_1, \ldots, h_s)$. A chain is of the first kind if its initial vertex is k_i and its terminal vertex is h_i. The sign attached to a chian of the first kind is $(-1)^{\ell+1}$ where ℓ is the length of the chain. A chain is of the second kind if its initial vertex is k_i and its terminal vertex is h_j, $j \neq i$. A chain of the second kind is not an element of a spanning set of cycles for L^*, but in general there will be elements of a spanning set of cycles for L^* which consist of the product of two or more chains of the second kind. The sign attached to such an element depends only upon the length ℓ of the product and is again $(-1)^{\ell+1}$. These rules make it a comparatively simple task to write out the formula for given specific minors of A .

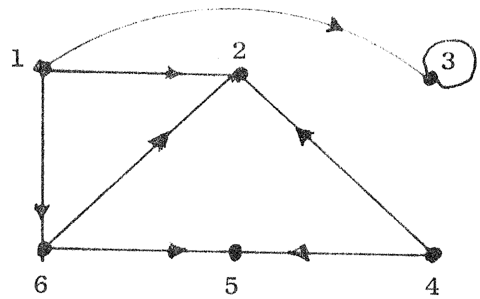
In view of the above discussion we have from formula (3) the result

$$A_{H,K} = (-1)^{\sum_{j=1}^{s}(\mu(h_j)+\mu(k_j))} \sum_{j=1}^{m} (-1)^{\nu_j} p_j A_{K_j} . \qquad (9)$$

Here, p_j is the product of all of the chains in the j-th spanning set of cycles of L^* in A , ν_j is the number of chains of the first kind or products of chains of the second kind of even length in the j-th spanning set, and A_{K_j} is the cominor of the j-th spanning set. Each A_{K_j} is a principal minor of $A(H \cap K)$ (which may be empty).

The factor $(-1)^{\sum_{j=1}^{s}(\mu(h_j)+\mu(k_j))}$ results from combining the sets of vertices $(h_1,\ldots,h_s)$ and $(k_1,\ldots,k_s)$ into the set of vertex pairs $(\langle k_1,h_1\rangle,\ldots,\langle k_s,h_s\rangle)$. This operation is equivalent to rearranging the rows and columns of $A(H,K)$ so that rows $(k_1,\ldots,k_s)$ are the first s rows and $(h_1,\ldots,h_s)$ are the first s columns.

<u>Example 8</u>. Consider again a matrix having the form illustrated in Example 6. Let $H = (1346)$, $K = (2356)$. Then $L = H \cup K = (123456)$, $L - H = (25)$, $L - K = (14)$, $\mu(1) = \mu(2) = 0$, $\mu(4) = \mu(5) = 1$. We obtain for $D(L)$ the graph shown in figure 5(a) and for $D_0(L)$ the graph shown in figure 5(b).



5(a)

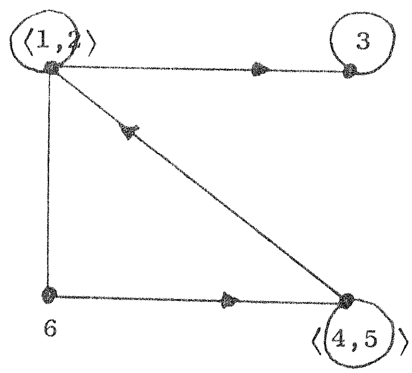


Figure 5(b)

Thus we have

$$A_{1346,2356} = a_{12}a_{45}a_{33}a_{66} + a_{16}a_{65}a_{42}a_{33}$$

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