COMBINATORIALLY SYMMETRIC MATRICES

John S. Maybee*

Department of Mathematics
Computer Science Department
University of Colorado
Boulder, Colorado

TR #CU-CS-021-73 July, 1973

* This research was supported by a grant from the National Science Foundation.

1. Introduction. The purpose of this paper is to collect together several results on an interesting class of matrices. Some of these have already appeared either implicitly or explicitly in the literature and others are new. It will moreover be clear that our results represent only a beginning and that many interesting and challenging problems remain.

In recent years a widely used tool in the study of matrices is graph theory. Unfortunately graph theory is an area in which no two authors seem to use the same terminology. We shall attempt to adhere to the terminology of the text by Harary [1] but will be forced to deviate occasionally in order to use terms previously used in matrix theory.

Given a matrix $A = [a_{ij}]_1^n$ we associate with it a digraph D(A) containing n points and a directed line from point i to point j iff $a_{ij} \neq 0$, $i \neq j$. Paths and cycles are defined in D(A) in the usual way. These give rise to corresponding concepts in A which we call chains and cycles respectively. Namely, if $i_1 i_2 \ldots i_p$ is a path in D(A) the corresponding chain in A is $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{p-1} i_p}$, and if $i_1 i_2 \ldots i_p i_1$ is a cycle in D(A) the corresponding cycle in A is $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{p-1} i_p} a_{i_p i_1}$. (These concepts are explained carefully in [3].)

Definition 1. The matrix $A = [a_{ij}]_1^n$ is combinatorially symmetric if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$.

Observe that if A is combinatorially symmetric, each line of D(A) is in a 2-cycle and D(A) can be regarded as a (undirected) graph G(A). As far as we can determine the first author to make significant use of this concept to study important problems related to the combinatorially symmetric case was Parter in the paper [2].

Each line of $G(A)$ corresponds to a nonzero 2-cycle of A. It follows that each cycle of $G(A)$ corresponds to a nonzero cycle of A of length greater than 2. We can go further. Suppose $\hat{a}(J) = a_{j_1 j_2} \dots a_{j_r j_1}$ is an r-cycle of A, $r > 2$. We denote by $\hat{a}^t(J)$ the transposed cycle $\hat{a}^t(J) = a_{j_2 j_1} \dots a_{j_1 j_r}$. If A is combinatorially symmetric, then $\hat{a}(J) \neq 0$ if $\hat{a}^t(J) \neq 0$. Thus each cycle of $G(A)$ actually corresponds to a pair of (transposed) nonzero cycles of A of length greater than 2.

Let $A = [a_{ij}]_1^n$ be an arbitrary matrix. Then A is indecomposable iff $D(A)$ is strongly connected. If A is combinatorially symmetric it is indecomposable iff $G(A)$ is connected. This means that in the case where A is combinatorially symmetric and decomposable there exists a permutation matrix P such that

$$P^t A P = A_1 \oplus \dots \oplus A_p$$

where each of the summands is itself indecomposable. It follows that we may essentially limit our discussion of such matrices to the class of indecomposable matrices. Accordingly we shall assume that combinatorial symmetry implies indecomposability.

In view of the fact that $G(A)$ does not depend upon the elements on the principal diagonal of A the following ideas seem natural. We shall denote by A_d the matrix with elements α_{jk} where $\alpha_{jj} = a_{jj}$,

$1 \leq j \leq n$, and $\alpha_{jk} = 0$ if $j \neq k$. We then set $\widetilde{A} = A - A_d$. Obviously $G(\widetilde{A}) = G(A)$. (The same concept may be defined relative to A in the general case and one again has $D(A) = D(\widetilde{A})$.)

2. Some interesting examples. We start with a definition.

Definition 2. The matrix $A = [a_{ij}]_1^n$ is an element of the class $\mathcal{J}_p$ if every r-cycle of A of length $r > p$ is zero and A has at least one nonzero p-cycle.

It is clear that these classes are of interest primarily in the case where p is small relative to n. The class $\mathcal{J}_1$ consists of, among others, diagonal matrices and upper triangular matrices and every element of this class is decomposable. The class $\mathcal{J}_2$ has been the object of a considerable amount of study and the main results have been summarized in the paper [4]. It is easy to give examples of matrices in $\mathcal{J}_2$ which are not combinatorially symmetric. However, let us introduce the class Q_2 defined by

$$Q_2 = \{A \in \mathcal{J}_2 \mid A \text{ is indecomposable}\}.$$

One can then prove

Theorem 1. $A \in Q_2$ iff A is combinatorially symmetric and $G(A)$ is a tree.

We will omit the proof of theorem 1. The only if portion is well-known and appears, for example, in the paper [5]. The if portion follows from a single graph theoretic argument.

Consider now the class $\mathcal{J}_3$. Unlike the class $\mathcal{J}_2$ an element of $\mathcal{J}_3$ can be indecomposable and not combinatorially symmetric. An example is given by a matrix of the form

$$A = \begin{bmatrix} x & x & x & 0 & 0 \\ x & x & x & x & 0 \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

where an "x" denotes a nonzero element. The digraph is

(The graph appears separately at the end of the manuscript.)

On the other hand, combinatorial symmetry is connected in a natural way with the class $\mathcal{J}_2$. It therefore seems reasonable to look for some natural connection between this concept and the class $\mathcal{J}_3$. One way to do this is to define a semi-bridge in a strongly connected digraph D to be a directed line whose removal will cause D to be no longer strongly connected. Thus a semi-bridge in the matrix A is an alement whose removal will cause A to become decomposable. For any matrix of the form displayed above the elements a_{21}, a_{32}, a_{43}, and a_{54} are all semi-bridges. We now define the class Q_3 by

$$Q_3 = \{A \in \mathcal{J}_3 | A \text{ is indecomposable and no 3-cycle contains a semi-bridge}\}.$$

<u>Theorem 2</u>. $A \in Q_3$ iff A is combinatorially symmetric.

<u>Proof</u>. Suppose first that $A \in \mathcal{J}_3$ is combinatorially symmetric. Since every line of G(A) corresponds to a 2-cycle of A it is clear that no 3-cycle can contain a semi-bridge. Hence $A \in Q_3$. For the converse suppose $A \in Q_3$ and consider D(A). Every directed line ℓ of D(A) either belongs to a (directed) triangle of D(A) or it does not. Suppose ℓ does not belong to a triangle and has the form (p,q). Then if (q,p) does not belong to D(A), D(A) is not strongly connected. Hence every such line belongs to a 2-cycle of D(A). We will complete the proof of the theorem by showing that if $A \in \mathcal{J}_3$ is indecomposable and is not combinatorially symmetric, then

some 3-cycle contains a semi-bridge. Let $a_{\alpha\beta}$, $\alpha \neq \beta$, be such that $a_{\beta\alpha} = 0$ for $A \in \mathcal{Q}_3$. There exists a nonzero chain $a(\beta \to \alpha)$ in A. Since A is indecomposable and $a_{\beta\alpha} = 0$, there exists an index i different from α and β such that $a(\beta \to \alpha) = a_{\beta i} a_{i\alpha}$. We claim that either $a_{i\alpha}$ or $a_{\alpha\beta}$ is a semi-bridge. In fact, suppose $a_{\alpha\beta}$ is not a semi-bridge. Then there exists a nonzero chain $a_1(\alpha \to \beta)$ in A distinct from $a_{\alpha\beta}$. Since $A \in \mathcal{Q}$ $a_1(\alpha \to \beta) = a_{\alpha i} a_{i\beta}$. Now for $a_{i\alpha}$ not to be a semi-bridge there must exist j distinct from i, α, and β such that $a_{\beta j} a_{j\alpha} \neq 0$. But then $a_{\alpha i} a_{i\beta} a_{\beta j} a_{j\alpha} \neq 0$ contradicting the fact that $A \in \mathcal{Q}_3$. Thus either $a_{\alpha\beta}$ or $a_{i\alpha}$ is a semi-bridge and the theorem is proved.

We remark here that the referee, who did an unusually conscientious job which the author appreciates, pointed out the following interesting facts about elements of $\mathcal{Q}_3$. If $A \in Q_3$ each block of $G(A)$ must be a line or a triangle of $G(A)$. Moreover, if X is a subset of vertices with $|X| \geq 4$ and if the vertex induced subgraph induced by X is connected, then X must contain at least one cutpoint.

It would be interesting to examine the properties of combinatorially symmetric matrices of $\mathcal{Q}_p$ for $p > 3$. This problem is open.

3. A theorem on symmetrization. A natural question which arises is the following. Suppose the real matrix A is combinatorially symmetric. When does there exist a nonsingular matrix S such that $S^{-1}\tilde{A}S$ is symmetric? This question has been answered in the case where S is a diagonal matrix, but no work seems to have been done in other cases.

We shall give now a slightly sharpened version of the theorem of Parter and Youngs [6].

Theorem 3. Let $A = [a_{ij}]_1^n$ be a real combinatorially symmetric matrix. Then there exists a real diagonal matrix D such that $D^{-1}\tilde{A}D$ is symmetric iff:

(i) There is a spanning tree T of $G(A)$ such that the 2 - cycles of A corresponding to the edges of T are all positive.

(ii) If $\hat{a}(J)$ is an r - cycle $(r > 2)$ of A corresponding to a chord of T and $\hat{a}^t(J)$ is the transposed r - cycle, then $\hat{a}(J) = \hat{a}^t(J)$.

Proof. Set $B = D^{-1}\tilde{A}D$ where $D = \text{diag}(d_1,\ldots,d_n)$, then $B = [b_{jk}]_1^n$ with $b_{jk} = d_k a_{jk}/d_j$. We require that for each $1 \le \beta < \alpha \le n$ the equality $b_{\alpha\beta} = b_{\beta\alpha}$ is satisfied if $a_{\alpha\beta} \ne 0$. Thus we must have

$$d_\beta a_{\alpha\beta}/d_\alpha = d_\alpha a_{\beta\alpha}/d_\beta . \qquad (1)$$

Since A is indecomposable, $G(A)$ is connected and hence contains at least one spanning tree T. Relation (1) must hold for the 2 - cycles corresponding to the $n-1$ edges of T. Now (1) implies that

$$d_\beta^2/d_\alpha^2 = \frac{a_{\beta\alpha}}{a_{\alpha\beta}} = \frac{a_{\alpha\beta}a_{\beta\alpha}}{a_{\alpha\beta}^2}$$

showing that if the elements of D are to be real we must have $a_{\alpha\beta}a_{\beta\alpha} > 0$ for these $n-1$ 2 - cycles. Next let b be a chord of T and suppose the adjunction of b to T results in the cycle c of G(A) with vertices $v_{i_1},\ldots,v_{i_p}$ where $b = [v_{i_1}, v_{i_p}]$, corresponding to the branch $[v_{i_1}, v_{i_2}]$ of T we have

$$d_{i_2}^2 = a_{i_2 i_1} d_{i_1}^2 / a_{i_1 i_2} ;$$

to the branch $[v_{i_2}, v_{i_3}]$ of T we have

$$d_{i_3}^2 = a_{i_3 i_2} d_{i_2}^2 / a_{i_2 i_3} ;$$

etc. Combining the formulas obtained in this way leads to

$$d_{i_1}^2 = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{p-1} i_p} d_{i_p}^2 / a_{i_2 i_1} \cdots a_{i_p i_{p-1}} .$$

On the other hand, from (1) applied to the edge b of G(A) we have

$$d_{i_1}^2 = a_{i_1 i_p} d_{i_p}^2 / a_{i_p i_1} .$$

Equating the two expressions for $d_{i_1}^2$ and cancelling the common factor $d_{i_p}^2$ from both sides leads to

$$a_{i_1 i_2} \cdots a_{i_{p-1} i_p} a_{i_p i_1} = a_{i_2 i_1} \cdots a_{i_p i_{p-1}} a_{i_1 i_p} .$$

Thus (ii) is proved and we have established the validity of the only if portion of the theorem.

To prove the if portion suppose (i) and (ii) are satisfied and let D be a diagonal matrix. The $n-1$ relations of the form (i) corresponding to the edges of T permit the determination of the elements of D up to a constant multiple. On the other hand, condition (ii) clearly implies the validity of (1) also for every chord of T, hence A is symmetrized by D.

Now let us agree to call A sign symmetric if $\operatorname{sgn} A$ is a symmetric matrix (see [3] or [6] for this concept), and let us call A pseudo symmetric if the conditions of the theorem are satisfied. Then the theorem of Parter and Youngs is

Theorem 3′ (Parter and Youngs). Suppose A is combinatorially symmetric. Then A is pseudo symmetric iff A is sign symmetric and $\hat{a}(J) = \hat{a}^t(J)$ for every pair of nonzero transposed r-cycles, $r > 2$, of A.

The advantage of the formulation given in theorem 3 over that in theorem 3′ is easily illustrated. Consider, for example, a full matrix A of order $n = 4$. A spanning tree has 3 edges and 3 chords. Thus only 3 pairs of transposed cycles need be checked for equality in order to establish pseudo-symmetry. On the other hand the total number of pairs of transposed cycles of length greater than 2 is 7. The difference is even more striking for larger values of n. It should be pointed out, however, that it is shown in [6] that it is only necessary to deal with a spanning tree of the graph.

4. Pseudo skew-symmetric matrices and their applications. We turn next to a theorem which is similar to theorem 3 relating to skew-symmetry.

Theorem 4. Let $A = [a_{jk}]_1^n$ be real and combinatorially symmetric. Then there exists a real diagonal matrix D such that $D^{-1}\tilde{A}D$ is skew-symmetric iff:

(i′) There is a spanning tree T of G(A) such that the 2-cycles of A corresponding to the edges of T are all negative.

(ii′) If $\hat{a}(J)$ is an r-cycle $(r > 2)$ of A corresponding to a chord of T and $\hat{a}^t(J)$ is the transposed r-cycle, then $\hat{a}(J) = (-1)^r \hat{a}^t(J)$.

Proof. We shall omit the proof of theorem 4 since it is entirely similar to the proof of theorem 3. A different version of theorem 4 appears in the paper of Parter and Youngs [6].

Let us agree to call a matrix A satisfying the conditions of theorem 4 a pseudo skew-symmetric matrix and denote the class of such matrices by $\mathcal{K}$.

The class $\mathcal{K}$ has interesting connections with the classes of sign stable and potentially stable matrices. We remind the reader that a matrix A is sign stable if every matrix B for which sgn B = sgn A is stable (all eigenvalues of B have negative real parts). On the other hand A is potentially stable if some matrix B such that sgn B = sgn A is stable. The class of sign stable matrices has been characterized by Quirk and Ruppert (see [3] and below) and it is known that every sign stable matrix is an element of the class $\mathcal{K}$. The problem of characterizing the class of potentially stable matrices is still open.

Define the classes $\mathcal{K}^+$ and $\mathcal{K}^-$ by

$$\mathcal{K}^+ = \{A \in \mathcal{K} | A_d > 0\}, \ \mathcal{K}^- = \{A \in \mathcal{K} | A_d < 0\} .$$

Theorem 5. If $A \in \mathcal{K}^+$ then the spectrum of A is contained in the open right half of the complex plane and if $A \in \mathcal{K}^-$ the spectrum of A is contained in the open left half of the complex plane, i.e., A is a stable matrix.

Proof. Let $A \in \mathcal{K}$ and suppose D is a diagonal matrix which skew-symmetrizes $\tilde{A}$. We then have

$$A' = D^{-1}AD = A_d + S$$

where S is skew-symmetric. Let λ be an eigenvalue of A' and $u \neq 0$ a corresponding eigenvector so that

$$(A_d + S)u = \lambda u .$$

In general u and λ will be complex. We use $u \cdot v$ to denote the standard complex scalar product and set $|u| = \sqrt{u \cdot u}$. Then

$$u \cdot A_d u + u \cdot Su = \bar{\lambda}|u|^2 ,$$

($\bar{\lambda}$ is the complex conjugate of λ). Also

$$A_d u \cdot u + Su \cdot u = \lambda |u|^2 .$$

Since $u \cdot A_d u = A_d u \cdot u$ and $u \cdot Su = -Su \cdot u$, we obtain

$$R(\lambda) = \frac{A_d u \cdot u}{|u|^2} \qquad (2)$$

where $R(\lambda) = \frac{1}{2}(\lambda + \bar{\lambda})$. Theorem 5 follows at once from (2).

We remark that theorem 5, which is a simple consequence of theorem 4, can be used to replace the use of the Liapunov theorem in the previously given proofs of the Quirk-Ruppert theorem on sign stable matrices (see [3] for a statement and proof of this theorem).

Moreover, the remainder of the proof of the sufficiency of the conditions in the Quirk-Ruppert theorem can be applied to obtain the following result.

Let us introduce the class $\mathcal{K}_1^-$ by

$$\mathcal{K}_1^- = \{A \in \mathcal{K} \mid \text{(i) } a_{ii} \leq 0 \text{ for } 1 \leq i \leq n \text{ and } a_{ii} < 0 \text{ for at least one } i,$$
$$\text{(ii) all nonzero even cycles of } A \text{ are negative,}$$
$$\text{(iii) determinant } A \neq 0\} .$$

This class is very closely related to the class of sign stable matrices which we shall denote by Q_2^- for present purposes:

$$Q_2^- = \mathcal{K}_i^- \cap \mathcal{J}_2 .$$

<u>Theorem 6</u>. Let $A \in \mathcal{K}_i^-$. Then A is potentially stable.

<u>Proof</u>. Since A is combinatorially symmetric, transposed p-cycles cancel out in pairs for p odd in the fundamental determinant formula of [3]. Thus, because of condition (ii), every term entering into the expansion of a principal minor of $A \in \mathcal{K}_1^-$ of order p will have sign $(-1)^p$ or zero. It follows that if $A \in \mathcal{K}_1^-$ every principal minor of A of order p has sign $(-1)^p$, $1 \leq p \leq n$. On the other hand, a combinatorial argument based upon conditions (i) and (iii) shows that A has a nested sequence of nonvanishing principal minors. The proof of this fact given in [3] for the subclass Q_2^- is in fact also valid for the entire class $\bar{\mathcal{K}}_1$. It follows by a theorem of Fisher and Fuller (see [3]) that there exists a positive diagonal matrix D such that

DA is a stable matrix. But, if $A \in K_1^-$ then $DA \in K_1^-$, proving the potential stability of A.

This theorem extends the class of potentially stable matrices considerably beyond previously known classes of such matrices.

The condition $d(A) \neq 0$ of theorem 6 is usually regarded as a qualitative condition, but its implications concerning the matrix A are complicated. This condition also occurs in the Quirk-Ruppert theorem.

We conclude with two examples illustrating the diverse implications of the hypothesis $d(A) \neq 0$.

<u>Example 1</u>. Consider the bordered diagonal matrix of order $n+1$

$$A = \begin{bmatrix} -a & b_1 & b_2 & \cdots & b_n \\ -b_1 & -c_1 & 0 & \cdots & 0 \\ -b_2 & 0 & -c_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_n & 0 & 0 & \cdots & -c_n \end{bmatrix}$$

where $a \geq 0$, $c_j \geq 0$, $1 \leq j \leq n$, and the b_j can be either positive or negative. Obviously $A \in K_1^-$ if $d(A) \neq 0$ and at least one diagonal element is negative. Now

$$d(A) = (-1)^{n+1} a \prod_{j=1}^{n} c_j + (-1)^{n-1} \sum_{j=1}^{n} b_j^2 \prod_{k \neq j} c_k .$$

Thus $d(A) = 0$ if 2 or more of the $c_j = 0$. In other words, the conditions of theorem 6 are met iff at least $n-1$ of the c_j are different from zero.

Example 2. Consider next the matrix B of order n having the form

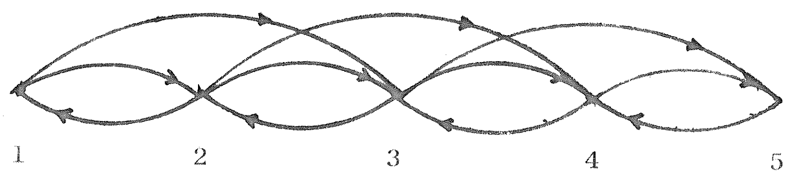
$$B = \begin{bmatrix} -a & b & 0 & \dots & 0 & 0 \\ -b & 0 & b & \dots & 0 & 0 \\ 0 & -b & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -b & 0 \end{bmatrix},$$

$a > 0$, $b \neq 0$. $B \in K_1^-$ if $d(B) \neq 0$. It is easy to verify by induction that

$$d(B) = b^{2p} \quad \text{if} \quad n = 2p .$$

$$d(B) = -ab^{2p} \quad \text{if} \quad n = 2p+1 .$$

Therefore the conditions of the theorem are met with only one nonzero element on the principal diagonal of B .


1
2
3
4
5

Bibliography

1. Harary, Frank, Graph Theory, Addison-Wesley, Reading, Mass., Menlo Park, Calif., London, Don Mills, Ontario, 1969.

2. Parter, Seymour V. On the eigenvalues and eigenvectors of a class of matrices. J. Soc. Ind. Appl. Math. 8, 376-388 (1960).

3. Maybee, J. S. and Quirk, J. P. Qualitative problems in matrix theory. SIAM Review 11, 30-51 (1969).

4. Maybee, J. S. Matrices of class J_2. J. Res. Nat. Bur. Standards, Sect. B, 71, 215-224 (1967).

5. Maybee, J. S. New generalizations of Jacobi matrices. SIAM J. Appl. Math., 14, 1032-1037 (1966).

6. Parter, Seymour V. and Youngs, J. W. T. The symmetrization of matrices by diagonal matrices. J. Math. Analysis and Applications, 4, 102-110 (1962).