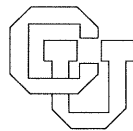


Approximate Linear Realizations Of Given Dimensions Via Ho's Algorithm

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ABSTRACT

Ho's Algorithm generates exact realizations of finite-dimensional linear systems given exact data. We discuss here a variant of the algorithm that generates approximate realizations of specified dimension from approximate data.

SECTION 0: INTRODUCTION

Ho's algorithm for identification of linear systems, together with some of its variants, has been viewed with justified suspicion by potential users who have only noisy data to work from. The principal difficulty is that noisy data introduces uncertainty about the rank of the Hankel matrix and hence about the dimension of resulting realization. Some time ago the senior author noticed that the ultimate weapon in such a situation would be an algorithm that produces a realization of given dimension that most closely approximates a given behavior. We present in this paper a promising start on the difficult problem of approximating a given behavior with a linear system of given dimension.

For simplicity, our algorithm is described for systems with scalar inputs and outputs, but this restriction is in no way essential.

SECTION 1: HO'S ALGORITHM

A linear, stationary, finite dimensional, dynamic system that operates in discrete time can be characterized by an $n \times n$ system of linear difference equations in the following form:¹

$$x_{t+1} = Ax_t + Bu_t$$

$$y_{t+1} = C^T x_{t+1}$$

The $n \times 1$ vector x_t represents the internal state of the system at time t , u_t is a scalar representing the input to the system at time t , and y_t is a scalar representing the output from the system at time t . The coefficient A is an $n \times n$ constant matrix, and B and C are $n \times 1$ constant vectors. The sequence $u_{t=0}, u_{t=1}, u_{t=2}, \dots$ of input values can itself be considered to be a vector U in a space of dimension equal to the length of the input sequence under consideration. Similarly, the sequence $y_{t=1}, y_{t=2}, y_{t=3}, \dots$ can be considered as a vector Y representing the output.

The same dynamical system, if controllable and observable, can also be characterized up to isomorphism by a vector -- the output vector Y which is the response to the unit input pulse, if there are enough time points included relative to the known bounds on the dimension of the system. [2] This characterization is easy to obtain, since all it requires are measurements of physical quantities. But the difference equations are often the desired form of representation. The problem is posed, then: Given a particular dynamical system and the response vector Y to a unit input pulse, how can one determine a set of coefficients

A, B and C necessary to characterize the system in the form of difference equations? First, it must be noted that the A, B and C are not in general unique for a given dynamical system. There may be many different dimensions. However, there will be a minimal dimension possible. Also, all systems of smallest possible dimension are identical up to an isomorphism, i.e., if A,B,C and A',B',C' are two sets of coefficients of dimension n which characterize the system, they are related by $A' = PAP^{-1}$, $B' = PB$, $C' = C^T P^{-1}$ for some nonsingular n x n matrix P [2]. Therefore, the eigenvalues of A are invariant -- any system of difference equations of smallest dimension will have an A with the same eigenvalues as the A in any other system of smallest dimension.

The problem can now be restated: How can one determine an A,B,C of minimal dimension to characterize the dynamical system in question? A solution to this problem was developed by B. L. Ho and is called "Ho's Algorithm." A description of the algorithm follows:

From any given unit pulse response, the minimal dimension of the system is not immediately apparent. A large number of elements of Y is used to insure that the dimension is not underestimated. If it is known (or guessed) that the system is not of dimension larger than some n, then 2n elements of the output vector Y are required. A matrix H is formed from these elements in the following manner:

$$H = \begin{bmatrix} y_1 & y_2 & y_3 & \cdot & \cdot & \cdot & y_n \\ y_2 & y_3 & y_4 & \cdot & \cdot & \cdot & y_{n+1} \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ y_n & y_{n+1} & y_{n+2} & \cdot & \cdot & \cdot & y_{2n-1} \end{bmatrix}$$

This matrix is then factored into the product of two matrices: $H = ST$, where S is $n \times m$ and T is $m \times n$. The inner dimension m is taken to be equal to the rank of H . Another matrix H' is formed by eliminating the first column of H and attaching the column:

$$\begin{matrix} y_{n+1} \\ y_{n+2} \\ \cdot \\ \cdot \\ \cdot \\ y_{2n} \end{matrix}$$

on the right side of H , so that we have:

$$H' = \begin{bmatrix} y_2 & y_3 & y_4 & \cdot & \cdot & \cdot & y_{n+1} \\ y_3 & y_4 & y_5 & \cdot & \cdot & \cdot & y_{n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_{n+1} & y_{n+2} & y_{n+3} & \cdot & \cdot & \cdot & y_{2^n} \end{bmatrix}$$

The equation $H' = SAT$ is then solved for A , the first row of S is taken to be C^T , and the first column of T to be B . Since H will have rank equal to the minimal dimension possible for A and the inner dimension m of the factorization is equal to the rank of H , A will be $m \times m$, and m will be the minimal dimension possible for the system. For any given factorization of H , the solution for A in this equation is both possible and unique as long as the inner dimension m is the rank of H [1], [2].

One of the drawbacks of this algorithm is that the results are often very sensitive to small changes in the data. This is undesirable especially since experimentally obtained data always contain some error. One of the problems which arises when using noisy data is that the rank of H may be increased. Therefore, the A matrix which one obtains from the algorithm may be of much larger dimension than the dynamical system actually warrants. Also, there may be numerical error mounting in the calculation of A in solving $H' = SAT$ if this is done on a finite precision computer. One would like to factor H into S and T in such a way that the numerical errors in solving for A are minimized. It would also be desirable to be able to reduce the rank of H if H has rank ρ but is

"close" to having rank $\eta < \rho$ as might occur with slightly perturbed elements of the matrix. Indeed, one highly-desirable procedure would be to choose in advance both a dimension m and a measure of goodness-of-fit to a given unit pulse response and somehow compute a realization A, B, C of dimension m whose unit pulse response best fits the given one. No algorithm for doing this now exists; if one were produced, it would sharply advance the state of linear modeling. This paper is devoted to a promising partial solution.

SECTION 2: TOOLS FOR DOING THE ALGORITHM APPROXIMATELY

Any matrix can be put into diagonal form under orthogonal equivalence. This is stated in the following theorem:

Theorem 1 - Singular Value Decomposition

Given any $n \times n$ real matrix H , there exist two $n \times n$ real orthogonal matrices U and V so that UHV^T is a diagonal matrix Q . U and V can be chosen so that the diagonal elements of Q are $q_1 \geq q_2 \geq \dots \geq q_r > q_{r+1} = \dots = q_n = 0$ where r is the rank of H . Thus if H is nonsingular, then $q_1 \geq q_2 \geq \dots \geq q_n > 0$. The elements q_i are called the singular values of H and are the positive square roots of the eigenvalues of HH^T . U consists of the n orthonormalized eigenvectors associated with the eigenvalues of HH^T . V consists of the n orthonormalized eigenvectors of H^TH [4].

For computational purposes, orthogonal matrices are, in general, more valuable than other types since $\|UX\|_2 = \|X\|_2$ for any orthogonal matrix U and any vector X . Therefore, multiplications by orthogonal matrices preserve the lengths of the vectors involved, whereas multiplications by a general nonsingular matrix may stretch or shrink the lengths, perhaps drastically. As a result, numerical errors are not amplified when orthogonal matrices are used.

The factorization of H in Ho's Algorithm can be done by singular value decomposition: $H = UQV^T$. Since the elements of Q are non-negative, this can be further factored as $H = U\sqrt{Q}\sqrt{Q}V^T$, where \sqrt{Q} represents the Q matrix with each element replaced by its square root. In general,

with noisy data and finite precision arithmetic in computing U , Q and V , H will have rank equal to its dimension. Therefore, Q will be non-singular even if H has larger dimension than the minimal one for the representation of the dynamical system. Hence one can easily solve for A in the following way:

$$A = \sqrt{Q}^{-1} U^T H V \sqrt{Q}^{-1}$$

since $U^{-1} = U^T$ and $(V^T)^{-1} = V$. \sqrt{Q}^{-1} is easy to compute since \sqrt{Q} is diagonal; one simply inverts each diagonal element.* The vectors B and C are obtained as described previously: C^T is the first row of $U\sqrt{Q}$ and B is the first column of $\sqrt{Q} V^T$.

We are still faced with the problem of reducing the dimension of A since the A obtained is of the same dimension as H . Again the singular value representation of H is helpful. Suppose we wish to approximate H with rank r by another matrix \hat{H} of rank $s < r$, and we wish $\|H - \hat{H}\|_2$ to be minimized so that \hat{H} will be "closest" to H . The following theorem will show that by setting the smallest nonzero element of Q in the singular value decomposition of H to zero, and then multiplying the matrices back together, the desired \hat{H} will be obtained.

Theorem 2

Let $H = UQV^T$ and choose $\hat{H} = U\hat{Q}V^T$ where $q_1 \geq q_2 \geq \dots \geq q_r > 0$ and $\hat{q}_1 = q_1, \hat{q}_2 = q_2, \dots, \hat{q}_s = q_s, \hat{q}_{s+1} = \dots = \hat{q}_r = 0$. Then \hat{H} is the closest matrix of rank s to H , i.e., $\|H - G\|_2$ is minimized for all G of rank s if $G = \hat{H}$.

* One must beware, however, of small elements of Q . There will be more discussion of this problem later.

We now show $\|H-G\| \geq q_{s+1}$ for an arbitrary G of rank s . Let G be any matrix of rank s . $\|H-G\|_2 = \|UQV^T - G\|_2 = \|Q = U^T G V\|_2$ since multiplying by an orthogonal matrix does not change the 2 norm. $U^T G V$ still has rank s , so we can look at $\|Q-F\|$ where $F = U^T G V$ without loss of generality. Assume $q_1 \geq \dots \geq q_r \geq q_{r+1} = \dots = q_n = 0$. The null space of F is of dimension $n - s$. The space spanned by e_1, e_2, \dots, e_{s+1} has dimension $s+1$. The intersection of these two spaces is nonempty, so we can choose a vector x which is in this intersection.

$$\|(Q-F)x\| = \|Qx - Fx\| = \|Qx\| = \left\| \begin{bmatrix} q_1 x_1 \\ q_2 x_2 \\ q_3 x_3 \\ \vdots \\ q_n x_n \end{bmatrix} \right\| =$$

$$= \left(\sum_{i=1}^n q_i^2 x_i^2 \right)^{1/2} = \left(\sum_{i=1}^r q_i^2 x_i^2 \right)^{1/2} \quad \text{since } x_1 \text{ through } x_{s+1}$$

are the only elements of x with nonzero values, we have $\left(\sum_{i=1}^r q_i^2 x_i^2 \right)^{1/2}$

$$= \left(\sum_{i=1}^{s+1} q_i^2 x_i^2 \right)^{1/2}. \quad \text{And since } q_i \geq q_{s+1} \text{ for } i < s+1,$$

$$\left(\sum_{i=1}^{s+1} q_i^2 x_i^2 \right)^{1/2} \geq q_{s+1} \left(\sum_{i=1}^{s+1} x_i^2 \right)^{1/2} = q_{s+1} \|x\|. \quad \text{So}$$

$$\|(Q-F)x\| \geq q_{s+1} \|x\| \quad \text{and} \quad \frac{\|(Q-F)x\|}{\|x\|} \geq q_{s+1}. \quad \text{But } \|Q-F\| =$$

$$= \max_{\|Y\| \neq 0} \frac{\|(Q-F)Y\|_2}{\|Y\|_2} - \frac{\|(Q-F)x\|_2}{\|x\|_2} \geq q_{s+1}. \text{ So } \|Q-F\|_2 \geq q_{s+1}$$

for any matrix of F of rank s. Therefore, the \hat{H} defined earlier is the closest matrix of rank s to H.*

When this theorem is applied to Ho's Algorithm, we are quite certain that the true dimension of the system must be less than n, but the H we get from noisy data has rank n. Therefore, we approximate H by \hat{H} of smaller rank. An A of similar smaller rank is obtained in the following way: We have $\hat{H} = U\hat{Q}V^T$, and we seek to determine A such that $\|H' - U\sqrt{\hat{Q}} A\sqrt{\hat{Q}} V^T\|_2$ is minimized. Now $\|H' - U\sqrt{\hat{Q}} A\sqrt{\hat{Q}} V^T\|_2 = \|U^T H' V - \sqrt{\hat{Q}} A\sqrt{\hat{Q}}\|_2$, and $\sqrt{\hat{Q}} A\sqrt{\hat{Q}}$ is zero in rows r+1, ..., n and columns r+1, ..., n. Thus the minimum will be assumed if $\sqrt{\hat{Q}} A\sqrt{\hat{Q}}$ equals $U^T H' V$ in the upper r x r submatrix. The remaining elements of A can be chosen arbitrarily, since the produce $\sqrt{\hat{Q}} A\sqrt{\hat{Q}}$ is independent of the components in rows and columns r+1, ..., n. Since an A is desired which is of smallest dimension possible, the last n - r rows and columns are chosen to be zero. This configuration indicates clearly the rank of A. Hence the equations for solution of the coefficients are: $A = \sqrt{\hat{Q}}^+ U^T H' V \sqrt{\hat{Q}}^+$ B = first column of $\sqrt{\hat{Q}} V^T$, C^T = first row of $U\sqrt{\hat{Q}}$, where $\sqrt{\hat{Q}}^+$ is the pseudo-inverse of $\sqrt{\hat{Q}}$ and is obtained by inverting each nonzero element of $\sqrt{\hat{Q}}$. Any other choice of an "inverse" for $\sqrt{\hat{Q}}$ which does not have zeros in the same positions of A. This choice of 0's is important in reducing the dimension of the system:

* We thank Alan Cline for supplying the critical step in this proof.

Consider the set of $n \times n$ difference equations created by the coefficients A with zeroes in the last $n - r$ rows and columns and B and C with $n - r$ zeroes in the last places. Also consider the set of $(n - r) \times (n - r)$ difference equations created by the same coefficients but with all the last rows and columns of 0's ignored and the dimensions shrunk. It is easy to verify that the unit pulse response of each of these systems will be the same. Therefore, the two systems are equivalent and the shrunken one can be chosen as the representation desired. In this manner a calculated A of smaller dimension of H can be obtained.

Singular value decomposition to factor H was chosen mainly for two reasons: orthogonal matrices are numerically more stable to work with than arbitrary matrices, and finding a matrix "close" to H but of smaller rank than H is easy. However, as indicated earlier, there is no proof at this time that singular value decomposition is the best method to use.

SECTION 3: WHAT TO EXPECT

As stated before, what we would like to do is derive a linear system whose unit pulse response sharply approximates the given one while simultaneously meeting a constraint on the state space dimension. Since the derived system will have a unit pulse response matrix $H=ST$, where S and T are of rank $r =$ desired dimension, a necessary condition for reaching our objective is to find a factorization ST through a space of dimension r that sharply approximates the H of the given unit pulse response. The version of Ho's Algorithm described in the preceding section meets this necessary condition as nearly as possible. What keeps the condition from being sufficient is that the proposed approximation may produce an A -matrix that is far enough off that the high powers of A required for the unit pulse might be grossly in error. One would expect this phenomenon to be worst for very unstable A 's, and least troublesome for stable A 's. Furthermore, in situations where this catastrophe does not occur, one would expect the pulse response error, when plotted against r , to have a minimum for intermediate values of r : When r is too small, we cannot approximate well because we do not have enough state space dimensions. When r is too large, small q 's appear and are inverted, upsetting the calculation.

The above expectations are difficult to check analytically because of the difficulty of estimating the errors in high powers of A . Experimental results, however, have fulfilled these expectations.

SECTION 4: EXPERIMENTAL TECHNIQUE AND RESULTS

To get some idea what might in practice be expected from the algorithm we ran the following experiments:

First, data was generated by simulating a chosen system of small dimension (usually three), the first ten to fourteen values of the unit pulse response were generated, and random numbers of mean equal to some chosen percentage of the maximum unit pulse response value were added. This procedure yielded data for which a three-dimensional approximation within, say, five percent was guaranteed possible. We then applied Ho's Algorithm with singular value decomposition to this data, truncating the Q matrix at various points to give realizations of various dimensions. The merit of each resulting realization was judged by measuring the percentage error in the worst pulse response value, and by comparing the natural frequencies of the computed realization with those of the original system.

The first striking result was that if the natural frequencies of the original system lay outside the unit disc (unstable system), then the computed realization was always grossly in error. A typical run exhibiting this phenomenon is shown in Figure 1.

(A belongs to the original system, A' $n \times n$ to the n -dimensional computed realization).

When the natural frequencies of the original system were all inside the unit disc (stable system), then much closer approximations resulted, provided the Q matrix was truncated at the right point. A typical run illustrating this situation is shown in Figure 2.

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\|Y\|_{\infty} = 512.$$

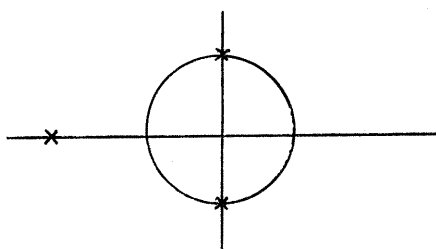
original Y vector

(5.00, -2.00, 0.00, =8.00, 20.0, -32.0, 60.0, =128., 260., =512.)

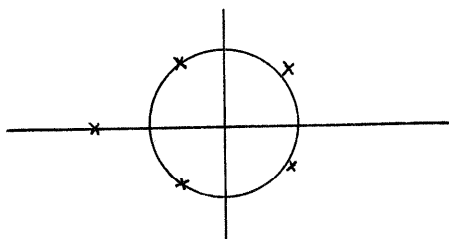
Q vector

(370, 42.9, 34.5, 34.0, 18.2)

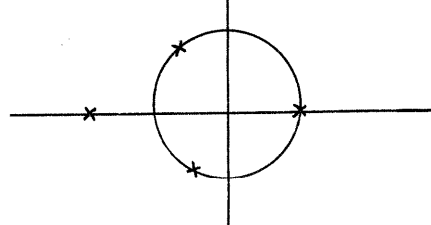
eigenvalues of A
i, -i, -2



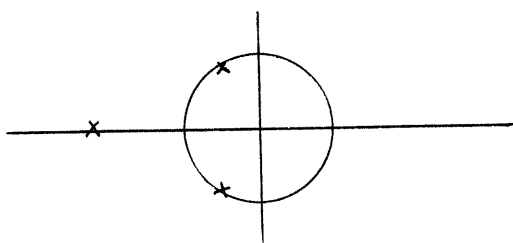
eigenvalues of A' 5x5
-1.8, $-.56 \pm .95i$, $.96 \pm .54i$
error $9.96 \times 10^{-12}\%$



eigenvalues of A' 4x4
-1.8, $-.56 \pm .93i$, 1.0
error 65.6%



eigenvalues of A' 3x3
-1.8, $-.41 \pm .89i$
error 65.6%



eigenvalues of A' 2x2
-.19, .24
error 279.0%

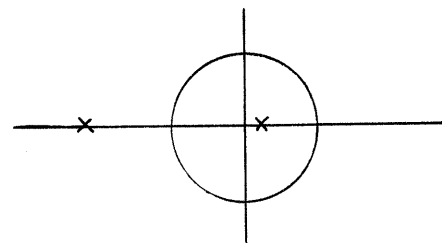


FIGURE 1

$$A = \begin{bmatrix} 0 & .7 & 0 \\ -.7 & 0 & 0 \\ 0 & 0 & -.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\|Y\|_{\infty} = 5.00$$

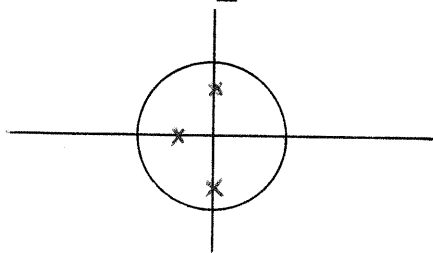
original Y vector

(5.00, .500, -1.71, -.125, 1.02, -.0312, -.456, -.00781, .234, -.00195, -.112, -.000488, .0556, -.000122)

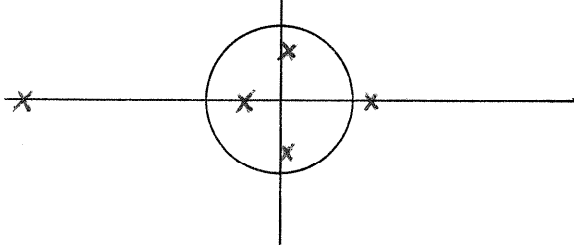
Q vector

(5.67, 2.32, .777, .382, .0605)

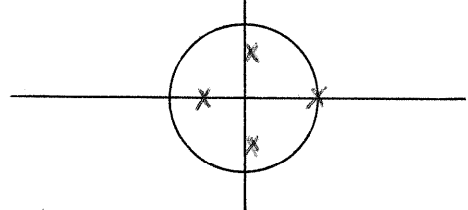
eigenvalues of A
-.50, ± .70i



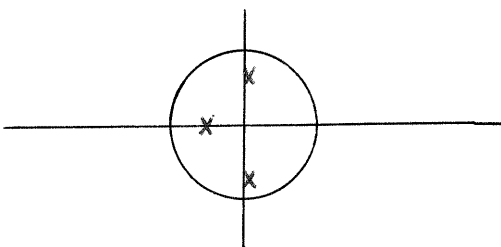
eigenvalues of A' 5x5
-3.1, -.57, 1.18, .034+.72i
error 4.7%



eigenvalues of A' 4x4
-.57, .98, .034+.72i
error 3.80%



eigenvalues of A' 3x3
-.55, .0032+.72i
error 4.34%



eigenvalues of A' 2x2
.053+.63i
error 8.76%

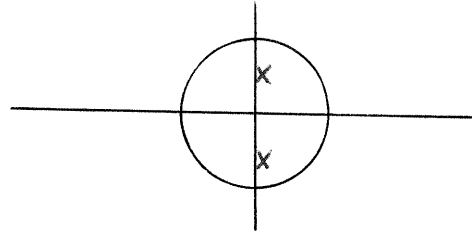


FIGURE 2

SECTION 5: CONCLUSIONS AND CHALLENGES TO THE READER

The experimental results summarized in the preceding section suggest that even the developmental version of Ho's Algorithm with Singular Value Decomposition described here is a practical tool for deriving approximate linear realizations of stable systems.

The following problems for further research suggest themselves:

1. Develop an error analysis of the algorithm described here.
2. Investigate variations on the above algorithm that might yield increased accuracy or ease of computation. (E.g., if we have enough data, we can shift H several columns instead of just one, set equal to SA^kT , and get powers of A directly. Can this information be put to use?)
3. How best can we code descriptions of continuous time systems for application of some variation of the given algorithm?
4. Compare (experimentally) the performance of this algorithm with that of J. Rissanen [3] in situations where both algorithms seem appropriate.

HPZ:cah

BIBLIOGRAPHY

- [1] R. E. Kalman, P. L. Falb, M. A. Arbib, Topics in Mathematical System Theory, McGraw-Hill, 1969.
- [2] H. P. Zeiger, "Ho's Algorithm, Commutative Diagrams, and the Uniqueness of Minimal Linear Systems," Information and Control 11, 71-79, (1967).
- [3] J. Rissanen, "Recursive Identification of Linear Systems," Privately communicated preprint.
- [4] G. Forsyth and C. Moler, Algebraic Systems, Prentice-Hall.