ENUMERATION OF NON-SEPARABLE GRAPHS ON FEWER THAN TEN POINTS *

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ABSTRACT:

Until recently, the number of non-separable graphs on seven or fewer points was known due to hand computations carried out by DeRocco [1]. By implementing an algorithm of Robinson [2] on a digital computer we were able to compute the number of non-separable graphs on nine or fewer points. This paper presents these numbers.

1. SOME BASIC CONCEPTS

In this paper we will be dealing exclusively with unlabeled graphs having no loops and no multiple edges, commonly called linear graphs.

Two points, a and b, of a graph are said to be adjacent if there is an edge of the graph having a and b as its end points.

The graphs G_1 and G_2 are isomorphic if there exists a one-to-one mapping of the points of G_1 onto the points of G_2 which preserves adjacencies. If no such mapping exists, then G_1 and G_2 are said to be nonisomorphic.

A point p of a connected graph G is said to be a cut point of G if G', the graph resulting from the removal of p and all edges having p as an end point, is not connected.

A non-separable graph is a connected graph having no cut points.

Throughout this paper, we shall denote the group of all permutations on n letters (known as the symmetric group) by S_n. If $\pi \varepsilon S_n$, then we define $pr(\pi)$ to be the permutation of pairs of letters induced by the letter permutation, π.

We shall also denote by R_n the group of all n! permutations of all $\frac{n(n-1)}{2}$ unordered pairs of n labelled points which are induced by permutations in S_n.

By the automorphism group of a graph, G, we shall mean the group of all permutations which map G onto itself.

The polynomials to which we shall refer in this paper may be polynomials in arbitrarily many variables, and may have an arbitrary number of terms. The coefficients of the terms will always be rational numbers. We shall make frequent use of the composition of two such polynomials, defined as follows: If $f(s_1,s_2,\ldots,s_i,\ldots)$ and $g(s_1,s_2,\ldots,s_i,\ldots)$ are polynomials in the variables $s_1,s_2,s_3,\ldots,s_i,\ldots$, we define their composition, $f(s_1,s_2,\ldots)\,[g(s_1,s_2,\ldots)]$, to be $f(g(s_1,s_2,\ldots s_i,\ldots), g(s_2, s_4,\ldots,s_{2i},\ldots), g(s_3,s_6,\ldots,s_{3i},\ldots),\ldots)$.

We shall have occasion to use the <u>cycle type</u> of a permutation, π, on n points. This is denoted by $Z(\pi)$, and defined to be the monomial in n variables, $s_1^{t_1}\, s_2^{t_2}\, s_3^{t_3} \ldots s_n^{t_n}$, where t_j is the number of cycles of length j in π.

Let us now consider a graph, G, and A(G), the group of all automorphisms of G (i.e., all permutations of the points of G which leave G fixed). The mean of the cycle types of all automorphisms of G, $\frac{1}{|A(G)|} \sum_{g \varepsilon A(G)} Z(g)$, shall be called the <u>cycle index</u> of the automorphism group of G, and will be denoted by $Z(A(G))$. Hence, for example, $\frac{1}{n!}\; Z(S_n)$ is the cycle index of the automorphism group of the graph on n points having no lines. In this paper, we shall be concerned largely with the computation of the sum of the cycle indices of the automorphism groups of all non-isomorphic graphs having a particular property.

We say that we can enumerate graphs having property P provided that for any n we can compute, by means of closed or recursive algebraic equations, the number of non-isomorphic graphs on i points having property P. In this connection, we shall use the term <u>counting polynomial</u>, defined as follows. The polynomial $p(x) = \sum_{i=1} a_i x^i$ is said to be the counting polynomial for graphs having property P provided that there are exactly a_i non-isomorphic graphs on i points having property P.

In general, we shall obey the following convention: If p is the counting polynomial for graphs having property P, then $\hat{p}$ will denote the sum of the cycle indices of all graphs having property P. We shall call $\hat{p}$ the cycle index polynomial for graphs having property P.

The method for enumerating non-separable graphs which is described here was developed in 1963 by R. W. Robinson in his senior thesis at Dartmouth College. It allows us to compute the cycle index polynomial for non-separable graphs. In order to do this, we first find the cycle index polynomial for linear graphs. Using Robinson's extension of Riddel's Equation, and the cycle index polynomial for linear graphs, we can obtain the cycle index polynomial for connected graphs. To this result, we can then apply a recursive technique developed by Robinson which yields the desired cycle index polynomial. The counting polynomial for non-separable graphs can then be derived by composing this cycle index polynomial over x.

Thus we see that in order to obtain the counting polynomial for non-separable graphs, it was first necessary to compute the cycle index polynomials for linear and connected graphs. By composing these polynomials over x we obtained counting polynomials for linear and connected graphs. These intermediate results, perhaps of interest in their own right, are included here also (see Table I).

2. ROBINSON'S METHOD

The following expositon merely states the important equations used in enumerating non-separable graphs. For detailed explanations of their derivations, the interested reader is referred to [2].

TABLE I

The Number of Connected, and Non-Separable Graphs on Nine or Fewer Points, and the Number of Linear Graphs on Eleven or Fewer Points

Number of Points	Number of Linear Graphs	Number of Connected Graphs	Number of Non-Separable Graphs
1	1	1	1
2	2	1	1
3	4	2	1
4	11	6	3
5	34	21	10
6	156	112	56
7	1,044	853	468
8	12,346	11,117	7,123
9	274,668	261,080	194,066
10	12,005,168		
11	1,018,997,864		

We can compute $\hat{Q}$, the cycle index polynomial for linear graphs from the equation:

2.1 $$\hat{Q} = \frac{1}{n!} \sum_{\pi \varepsilon S_n} (Z(pr(\pi)) \quad [2])\ Z(\pi)$$

Let us call a sequence of integers, $t_1, t_2, \ldots, t_n$, a partition of n if $\sum_{i=1}^{n} i\, t_i = n$. If $t_1, t_2, \ldots, t_n$ is a partition of n, then there are exactly $\frac{n!}{1^{t_1}\, t_1! \ldots n^{t_n}\, t_n!}$ permutations in S_n having cycle type $s_1^{t_1}\, s_2^{t_2} \ldots s_n^{t_n}$.

By brute force observation of special cases, we can compute that if $Z(\pi) = s_1^{t_1}\, s_2^{t_2} \ldots s_n^{t_n}$, then

2.2 $$Z(pr(\pi)) = \frac{n!}{1^{t_1} t_1! \ldots n^{t_n} t_n\,!} \left(\prod_{\substack{i \leq n \\ i\ \text{odd}}} s_i^{\,t_i\left(\frac{i-1}{2}\right)} \right) \left(\prod_{\substack{i \leq n \\ i\ \text{even}}} s_i^{\,t_i\,\left(\frac{i}{2} - 1\right)}\, s_{\frac{i}{2}}^{\,t_i} \right) \left(\prod_{i \leq n} s_i^{\,i t_i (t_i - 1)/2} \right) \left(\prod_{i < j \leq n} s_{[i,j]}^{\,t_i t_j (i,j)} \right)$$

where (a,b) is the greatest common divisor of a and b, and [a,b] is the least common multiple of a and b.

Hence it follows from 2.2 that if π_1 and π_2 are permutations having the same cycle type, then $Z(pr(\pi_1)) = Z(pr(\pi_2))$. If $t_1, t_2, \ldots, t_n$ is a partition of n, then we define $Y(t_1, t_2, \ldots t_n) = Z(pr(\pi))$, where π is any partition having cycle type $s_1^{t_1}\, s_2^{t_2} \ldots s_n^{t_n}$.

We can now write:

$$2.3 \quad \hat{Q} = \sum_{n=1}^{\infty} \sum_{\substack{\text{partitions} \\ \text{of } n, \\ t_1, t_2, \ldots t_n}} \frac{1}{1^{t_1} t_1! \ldots n^{t_n} t_n!} Y(t_1, \ldots, t_n)[2] s_1^{t_1} \ldots s_n^{t_n}.$$

Riddel's equation states that:

$$2.4 \quad \sum_{r=1}^{\infty} \frac{1}{r} q(x^r) = - \sum_{r=1} \frac{(-Q(x))^r}{r}$$

where $q(x)$ is the counting polynomial for connected graphs. Robinson has shown that Riddel's equation remains valid when $q(x^r)$ and $Q(x)$ are replaced by $s_r[\hat{q}]$ and $\hat{Q}$, respectively. Equivalently,

$$2.5 \quad \hat{Q} = \exp \left(\sum_{n=1}^{\infty} \frac{s_n[\hat{q}]}{n} \right) - 1$$

$$2.6 \quad = \sum_{n=1}^{\infty} \frac{s_n [\hat{q}]}{n} + \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{s_n [\hat{q}]}{n} \right)^2 + \frac{1}{6} \left(\sum_{n=1}^{\infty} \frac{s_n [\hat{q}]}{n} \right)^3 + \ldots .$$

Again, $\hat{q}$ can be determined recursively from $\hat{Q}$.

We can now define the operator, ', which operates on polynomials in n variables, $s_1, s_2, s_3, \ldots, s_n$, as follows:

$$p(s_1, s_2, s_3, \ldots, s_n)' = \frac{dp(s_1, s_2, s_3, \ldots, s_n)}{ds_1}$$

Let us now define $\hat{R} = \hat{q}'$. Let $\hat{B}$ be the sum of the cycle indices of the automorphism groups of all non-separable graphs. Then

$$2.7 \quad \hat{R} = \exp \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \; s_n [(\hat{B}' - 1) [s_1 \hat{R}]] \right) \right)$$

enabling us to solve recursively for $\hat{B}'$.

Moreover:

2.8 $\hat{q}|_{s_1=0} = (\hat{B}|_{s_1=0})\ [s_1\hat{R}]$ enabling us to determine $\hat{B}|_{s_1=0}$.

We now observe that:

2.9 $\hat{B} = {}_0\int^{s_1}\hat{B}\ '\ ds_1 + \hat{B}\ |s_1=0.$

Hence 2.7 and 2.8 suffice to determine $\hat{B}$. B(x), the counting polynomial for non-separable graphs is obtained by noting that

2.10 $B(x) = \hat{B}\ [x]$.

3. <u>THE COMPUTER PROGRAM</u>

The numbers in Table I were produced by implementing Robinson's method by means of a FORTRAN program on an IBM 7094 computer. The cycle index polynomials were represented and manipulated internally through the use of data packing and elementary list processing techniques. Numeric values were represented as integer pairs (numerator and denominator) in order to insure complete accuracy. Unfortunately, the representation of some of the intermediate numeric results of this method taxed the integer representation capacity of the 7094 machine word.

Hence we were unable to use this program to compute the number of non-separable graphs on ten points or more. It is clear that by representing numeric values differently and/or using a computer with larger word size we could extend the results tabulated here.

REFERENCES

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