SOME CLASSES OF UNIQUELY COLORABLE GRAPHS

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ABSTRACT

In this paper, we shall present the concept of a uniquely n-colorable graph, and then introduce a class of graphs which we shall call 6-clique rings. We shall show that 6-clique rings are useful in generating some classes of uniquely 3-colorable graphs. Moreover, we shall demonstrate how the techniques used in producing uniquely 3-colorable graphs from 6-clique rings can be extended to allow the production of other classes of uniquely 3-colorable graphs, as well as the production of uniquely n-colorable graphs, where n>3.

1. INTRODUCTION

Several workers have investigated the properties of uniquely n-colorable graphs (see for example [1], [2], and [4]). A number of necessary conditions for unique colorability have been found, as well as a number of sufficient conditions. A rather complete summary of these results can be found in Harary [3]. In this paper we study primarily the more specific problem of characterizing uniquely 3-colorable graphs. We are able to characterize a number of classes of uniquely 3-colorable graphs. The techniques used here seem applicable to the more general study of unique n-colorability in graphs.

2. SOME DEFINITIONS

In this paper we shall follow the notations and terminology of Harary [3]. The definitions of the more elementary concepts can also be found in Harary [3]. Note that we shall be concerned here only with finite unlabeled graphs having neither loops nor multiple edges.

It is first necessary to establish a few definitions.

Definition 2.1: A <u>k-coloring</u> of a graph, G, is a function, ϕ, which maps the vertices of G onto $\{1,2,\ldots,k\}$ in such a way that if v_1 and v_2, vertices of G, are adjacent, then $\phi(v_1)\neq\phi(v_2)$.

Definition 2.2: Let ϕ be a k-coloring of the graph G. The $\phi^{-1}(i)$, $1\leq i\leq k$, are called the <u>color classes</u> of G.

Definition 2.3: A <u>k-colorable</u> graph is a graph for which there exists a k-coloring.

Definition 2.4: The <u>chromatic number</u> of a graph, G, is the minimum n for which G is n-colorable. We shall denote the chromatic number of G by $\chi(G)$.

Definition 2.5: Two k-colorings of G, ϕ_1 and ϕ_2 are <u>equivalent</u> (we denote this by $\phi_1 \sim \phi_2$) if there exist τ, an automorphism of G, and ν, a k-permutation such that $\phi_1(\tau(v))=\nu(\phi_2(v))$ for all v, vertices of G.

Definition 2.6: G is <u>uniquely n-colorable</u> provided that $n=\chi(G)$ and that all n-colorings of G are equivalent.

In the following sections of this paper, we shall be most interested in graphs of the type which we now define.

Definition 2.7: A <u>6-clique ring</u> is a graph, G, consisting of six cliques which can be labelled $C_0, C_1, \ldots, C_5$ in such a way that no point of any C_i, $0 \le i \le 5$, is adjacent to any point of any C_j, $0 \le j \le 5$, unless i=j, $i-j \equiv 1 \pmod 6$, or $i-j \equiv 5 \pmod 6$.

If C_i is a clique of G, a six-clique ring, then we shall say that $C_{i+1 \pmod 6}$ and $C_{i+5 \pmod 6}$ are C_i's neighboring cliques or <u>neighbors</u>.

Hence, intuitively we can think of a six clique ring as being a graph consisting of six cliques arranged in a circle so that the points of a given clique are not connected to the points of any other clique unless that other clique is a neighbor.

Figure 1 is a schematic diagram of a 6-clique ring in which a clique is indicated by a circle(O), and neighboring cliques are connected by a line.

Figure 2 shows some examples of 6-clique rings. In Figure 2 we identify the cliques by surrounding them by dotted lines.

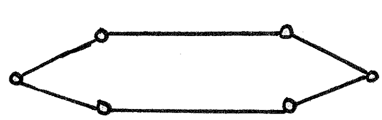
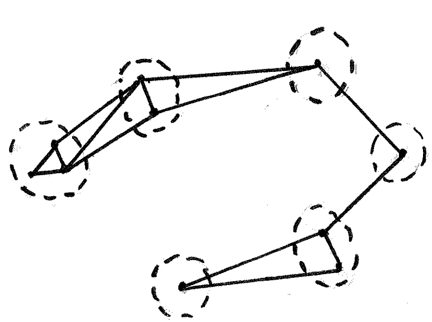
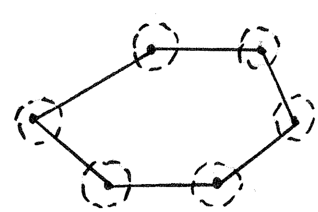


Fig. 1: Schematic representation of a six-clique ring



(2a.)

Fig. 2: Some examples of six-clique rings



(2b.)

It turns out that the complements of 6-clique rings have interesting colorability properties. In the following section we shall prove some results about the complements of 6-clique rings which will be useful in later sections.

3. SOME USEFUL LEMMAS ABOUT COMPLEMENTS OF SIX-CLIQUE RINGS

Lemma 3.1: If G is a six-clique ring, then $\chi(G')\geq 3$.

Proof: G' must clearly contain a triangle (K_3) as a subgraph. $\chi(K_3)=3$, thus $\chi(G')\geq 3$.

We have established three as a lower bound on the colorability of complements of 6-clique rings. We now set three as an upper bound also and study the structure of graphs having this property.

Lemma 3.2: Let G be a six-clique ring, and $\chi(G')=3$. Let ϕ be any 3-coloring of G'. Then for each C_i, a clique of G, ϕ must map all points of C_i onto the same color.

Proof: The proof is by contradiction. Suppose p_{i_1} and p_{i_2} are both points of C_i, $0\leq i\leq 5$. Suppose ϕ is a coloring of G' for which $\phi(p_{i_1})\neq\phi(p_{i_2})$.

Let $p_j\varepsilon C_{i+2 \pmod 6}$ and $p_k\varepsilon C_{i-2 \pmod 6}$. Clearly p_{i_1}, p_j, and p_k are the vertices of a triangle on G'. Hence p_{i_1},p_j, and p_k must have different colors. But p_{i_2},p_j, and p_k must also be the vertices of a triangle. Thus p_{i_2},p_j, and p_k have different colors. But $\phi(p_{i_1})\neq\phi(p_{i_2})$ by hypothesis. Thus p_{i_1},p_{i_2},p_j, and p_k all must have different colors. Hence ϕ cannot be a 3-coloring of G'.

We now see that when dealing with six-clique rings whose complements have chromatic number three it is reasonable to talk about the color of a clique. If C is a clique of G, a six clique ring, then we shall refer to the color onto which all points of C are mapped by ϕ, a 3-coloring of G', and the "color of C," and denote it by $\phi(C)$.

Lemma 3.3: Let G be a 6-clique ring for which $\chi(G')=3$. If two cliques of G have the same color, they must be adjacent cliques.

Proof: Cliques having the same color must be disjoint in G'. But non-adjacent cliques must be disjoint in G (by the definition of a six-clique ring), hence cannot be disjoint in G'. Hence non-adjacent cliques cannot have the same color in G'.

From Lemma 3.3 we can conclude that no color can color more than two beads of a six-clique ring whose complement is 3-colorable. But if no color can color more than two beads, we easily see that every color must color more than one bead. Thus we see that every color must color exactly two beads. By Lemma 3.3, moreover, we know that these beads must be adjacent. Since each clique has only two neighbors, we see there are exactly two possible colorations of G'. We have thus proved:

Lemma 3.4: Let G be a six-clique ring such that $\chi(G')=3$. If $C_0,C_1,\ldots,C_5$ are the six cliques of G, and ϕ is a 3-coloring of G', then either

(3.4.1) $\phi(C_0)=\phi(C_1)\neq\phi(C_2)=\phi(C_3)\neq\phi(C_4)=\phi(C_5)\neq\phi(C_0)$ or

(3.4.2) $\phi(C_1)=\phi(C_2)\neq\phi(C_3)=\phi(C_4)\neq\phi(C_5)=\phi(C_0)\neq\phi(C_1)$

It is easily seen that all 3-colorings, ϕ, satisfying (3.4.1) must be equivalent. Suppose ϕ_1, ϕ_2 are three-colorings satisfying (3.4.1), then $\exists\,\tau$, a 3-permutation, for which $\tau(\phi_1)=\phi_2$, since ϕ_1 and ϕ_2 can differ only in the colors onto which they map.

Similarly all ϕ satisfying (3.4.2) must be equivalent.

Hence we see that if G is a six-clique ring whose complement is three-colorable, then G' admits at most two non-equivalent three-colorings. If the two 3-colorings are not equivalent, then we shall say that G' is <u>bi-three-colorable</u>. If the two colorings are equivalent, then G' is uniquely three colorable.

We summarize the results of this section with

Theorem 3.5: If G is a six-clique ring such that $\chi(G')=3$, then G' is either bi-three-colorable or uniquely three-colorable.

4. SOME CLASSES OF UNIQUELY THREE-COLORABLE GRAPHS OBTAINABLE FROM COMPLEMENTS OF SIX-CLIQUE RINGS

We now attempt to determine the conditions under which the three-colorings of the complement of a six-clique ring are equivalent. Towards this end, the following concept is useful.

Definition 4.1: A complete six-clique ring on the clique $C_0,C_1,\ldots,C_5$, is the maximal six-clique ring on cliques $C_0,C_1,C_2,\ldots,C_5$.

We easily see that given six cliques, $C_0,C_1,C_2,\ldots,C_5$, the corresponding complete six-clique ring is the graph containing all six cliques, and all the complete bipartite graphs between neighboring cliques. Note that Figure 2.b is a complete six clique ring.

If G is a six-clique ring whose cliques are $C_0,C_1,\ldots,C_5$, we shall find it convenient to refer to G, the complete six clique ring whose cliques are $C_0,\ldots,C_5$, as the completion of G.

Figure 3 shows the completion of the six-clique ring pictured in Figure 2.a

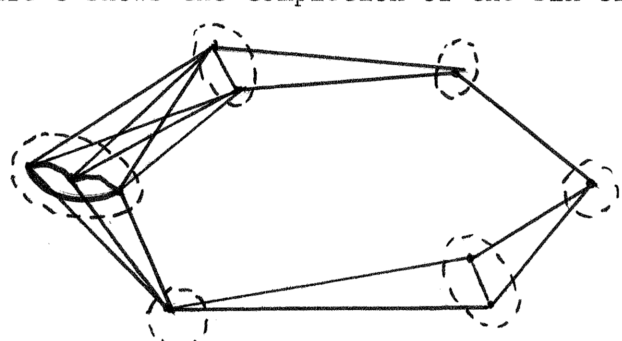


Fig. 3

Definition 4.2: Let G be a six clique ring. A thread, or non-clique line of G is a line of G whose endpoints lie in different cliques.

An i-thread is a thread whose endpoints lie in C_i and $C_{i+1(\text{mod } 6)}$.

Definition 4.3: A odd wad of a six-clique ring, G, is a collection of threads of G such that every thread is an i-thread where i is either 1 or 3 or 5.

An even wad of a six-clique ring, G, is a collection of threads of G such that every thread is an i-thread where i is either 0 or 2 or 4.

Theorem 4.4 Let H be a complete six-clique ring with cliques $C_0, C_1, \ldots, C_5$. If G is a six clique ring formed by the removal from H of any non-void odd wad, or any non-void even wad, then G' is uniquely three-colorable.

Proof: Let us examine G', the complement of a graph formed by deleting a non-void odd wad from H. Clearly C_i is disjoint in G' from $C_{i+1 \pmod 6}$ for i=0,2, and 4. There exists an i, however, i=1,3, or 5, for which C_i is not disjoint in G from $C_{i+1 \pmod 6}$.

Hence G' is three-colorable by a coloring, ϕ, satisfying (3.4.1), and G' is not three-colorable by a coloring ϕ, satisfying (3.4.2). Hence there is only one possible equivalence class of colorings of G'. Thus G' is uniquely three-colorable.

Exactly analogous methods suffice to prove that the removal of a non-void even wad from H produces a graph whose complement is uniquely three-colorable.

Corollary 4.5: Let H be a complete six-clique ring with cliques $C_0, C_1, \ldots, C_5$. If G is a six clique ring formed by the deletion of a non-void odd wad and a non-void even wad from H, then $\chi(G') > 3$.

Proof: Reasoning as in the proof of Theorem 4.4 we find that neither of the two possible equivalence classes of three-colorings specified in Lemma 3.4 can supply a three-coloring of G'.

We now turn our attention to the complements of complete six-clique rings. We wish to identify those classes of complete six-clique rings whose complements are uniquely three-colorable. The condition for unique colorability is stated in terms of the relative cardinalities of the various cliques. Hence we adopt the following notation:

We denote by $|C_i|$ the number of points in C_i (the cardinality of C_i).

Theorem 4.6: Let G be a complete six-clique ring with cliques $C_0,\ldots,C_5$. G' is uniquely three-colorable if and only if either:

(4.6.1) $|C_0| = |C_3|$, $|C_1| = |C_4|$, and $|C_2| = |C_5|$ or

(4.6.2) $|C_{i-1(\text{mod } 6)}| = |C_{i+1(\text{mod } 6)}|$ and $|C_{i-2(\text{mod } 6)}| = |C_{i+2(\text{mod } 6)}|$

for some i, $0\leq i\leq 5$

Proof: We prove the sufficiency first. Since G is a 6-clique ring, we can apply Lemma 3.4 and see that there are two (possibly equivalent) classes of three-colorings for G'. The sufficiency will be proved if we can show that these are equivalent.

Hence let us assume ϕ is a three-coloring of G' satisfying (3.4.1) and ψ is a three-coloring of G' satisfying (3.4.2). Without loss of generality, let us assume

$\phi(C_0)=\phi(C_1)=1;\ \phi(C_2)=\phi(C_3)=2;\ \phi(C_4)=\phi(C_5)=3$ and

$\psi(C_1)=\psi(C_2)=1;\ \psi(C_3)=\psi(C_4)=2;\ \psi(C_5)=\psi(C_6)=3$

By our definition of equivalence of colorings, we must now produce τ, an automorphism of G', and ν a three-permutation such that $\phi(\tau(v))=\nu(\psi(v))$ for all $v\varepsilon G'$.

Let us first assume that the cliques of G satisfy (4.6.1). We take τ to be any automorphism which interchanges the points of C_0 with the points of C_3, the points of C_1 with the points of C_4 and the points of C_2 with the points of C_5. We take ν to be (321). Equivalence is now easily verified.

If the cliques of G satisfy (4.6.2), we take τ to be any automorphism fixing the points of C_i and $C_{i+3(\text{mod } 6)}$, but interchanging the points of $C_{i+1(\text{mod } 6)}$ with the points of $C_{i-1(\text{mod } 6)}$ and the points of $C_{i+2(\text{mod } 6)}$ with the points of $C_{i-2(\text{mod } 6)}$. We take ν to be (3)(12). Again equivalence is easily verified.

In both cases, it is important to note that the automorphism τ can be found only if the cardinalities of the cliuqes are related as hypothesized.

We now prove the necessity. We assume that ϕ and ψ are three-colorings of G' satisfying:

$$\phi(C_0)=\phi(C_1)\neq\phi(C_2)=\phi(C_3)\neq\phi(C_4)=\phi(C_5)\neq\phi(C_0) \quad \text{and}$$

$$\psi(C_1)=\psi(C_2)\neq\psi(C_3)=\psi(C_4)\neq\psi(C_5)=\psi(C_0)\neq\psi(C_1)$$

Our hypothesis that G' is uniquely three-colorable means that ϕ and ψ are equivalent. Hence $\exists\,\tau$, an automorphism of G', and ν, a three-permutation, such that $\phi(\tau(v)) = \nu(\psi(v)), \forall v\varepsilon G'$.

Lemma 3.2 tells us that all points of a given clique of G must be mapped onto the same color by any three-coloring of G'. Hence we can state the following:

$$\phi(\tau(C_i)) = \nu(\psi(C_i)) \qquad i=0,1,\ldots,5$$

Since ν is a three-permutation, ν^{-1} exists and is also a three-permutation, thus

$$(4.6.3) \qquad \nu^{-1}(\phi(\tau(C_i))) = \psi(C_i) \qquad i=0,1,\ldots,5$$

We saw that ψ must map all the points of C_1 and C_2 onto the same color. Hence (4.6.3) says that $\nu^{-1}\circ\phi\circ\tau$ must also map all points of C_1 and C_2 onto the same color.

But ϕ maps C_1 and C_2 onto different colors. It is true, however, that ϕ maps the points of C_0 and C_1 onto the same color, the points of C_2 and C_3 onto the same color, and the points of C_4 and C_5 onto the same color. Hence τ must be an automorphism of G' which maps all the points of any given clique onto all the points of some (possibly different) clique. It is this condition that imposes the relations between cardinalities of cliques stated in (4.6.1) and (4.6.2). We develop these relations by examination of cases.

CASE I. The clique pair C_1,C_2 is mapped by τ onto the clique pair C_0,C_1. There are two subcases here; $\tau(C_1)=C_1$, $\tau(C_2)=C_0$ and $\tau(C_1)=C_0$, $\tau(C_2)=C_1$.

Suppose $\tau(C_1)=C_1$, $\tau(C_2)=C_0$. We are now interested in $\tau(C_3)$.

C_3 and C_2 are disjoint in G', hence $\tau(C_2)$ and $\tau(C_3)$ must also be disjoint in G'. But $\tau(C_2)=C_0$ and C_0 is disjoint in G' only from C_1 and C_5. Since $\tau(C_1)=C_1$ and τ is an automorphism, we must have $\tau(C_3)=C_5$. Since τ is an automorphism it is clear that $|C_2|=|C_0|$ and $|C_3|=|C_5|$. Hence (4.6.2) is satisfied for i=1.

Suppose $\tau(C_1)=C_0$, $\tau(C_2)=C_1$. As before, $\tau(C_3)$ would have to be disjoint in G' from $\tau(C_2)$ which is C_1 by hypothesis. $\tau(C_3)$ could not be C_0, thus $\tau(C_3)=C_2$. By similar reasoning, we would get $\tau(C_4)=C_3$, $\tau(C_5)=C_4$, $\tau(C_0)=C_5$. Hence we would conclude $|C_1|=|C_2|=|C_3|=|C_4|=|C_5|=|C_0|$, clearly satisfying both (4.6.1) and (4.6.2).

CASE II. The clique pair C_1,C_2 is mapped by τ onto the clique pair C_2,C_3.

It is easy to verify that this case is equivalent to case I.

CASE III. The clique pair C_1,C_2 is mapped by τ onto the clique pair C_4,C_5. There are two subcases here too; $\tau(C_1)=C_4$, $\tau(C_2)=C_5$ and $\tau(C_1)=C_5$, $\tau(C_2)=C_4$.

Suppose $\tau(C_1)=C_4$, $\tau(C_2)=C_5$. $\tau(C_3)$ must be disjoint in G' from $\tau(C_2)$, which is C_5. Hence $\tau(C_3)$ must be C_0 (since $\tau(C_1)=C_4$ and τ is an automorphism). Thus we must have $|C_1|=|C_4|$, $|C_2|=|C_5|$, and $|C_3|=|C_0|$, satisfying (4.6.1).

Suppose $\tau(C_1)=C_5$, $\tau(C_2)=C_4$. Then $|C_1|=|C_5|$ and $|C_2|=|C_4|$, which satisfied (4.6.2) for i=3.

Hence the necessity is proven.

5. SOME EXTENSIONS.

Until now we have identified certain classes of uniquely three-colorable graphs as being complements of certain six-clique rings. We now observe that there are other "clique structured" graphs whose complements have colorability properties very similar to the colorability properties of complements of six clique rings.

We use the schematic notation of Figure 1 to introduce three new families of graphs. As in Figure 1, a circle will be used to represent a clique. In these figures, however, circles which are joined by a line will denote cliques joined by the complete bipartite graph. Circles not joined by a line will denote disjoint cliques. We shall call these families of graphs the CG_7, CG_8, and CG_9 families.

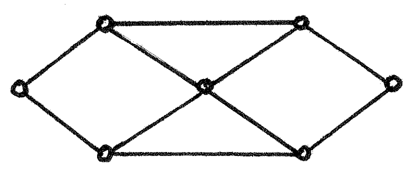


Fig. 4: Schematic representation of the CG_7 family of graphs

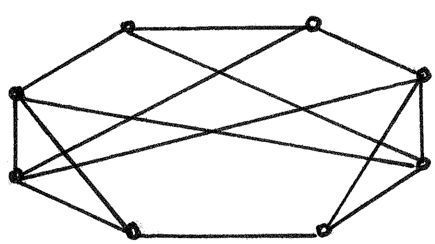


Fig. 5: Schematic representation of the CG_8 family of graphs

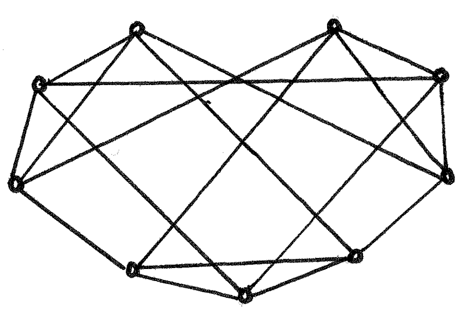


Fig. 6: Schematic representation of the CG_9 family of graphs

We assert that the methods of sections 3 and 4 are applicable, with minor modifications, to these three families of graphs. Hence families of uniquely three colorable graphs can be generated in similar ways. For example the following theorem can be proven by extending our six-clique ring results.

Theorem 5.1: Let G be a graph obtained by deleting a thread from a graph which is a member of CG_i, i=7, 8, or 9. Then G' is uniquely three-colorable.

Analogues to Theorem 4.4 can easily be generated for CG_i graphs (i=7,8,9) once the analogues to odd and even wads have been determined.

The analogues to Theorem 4.6 for CG_i graphs (i=7,8,9) promise to be harder to state and prove, but conceptually similar to what we have seen.

Finally, we observe that the techniques developed and used here seem applicable to the study of unique n-colorability for $n>3$. For example, we can prove by analogous methods that the complement of an 8-clique ring is either uniquely four colorable or bi-four-colorable, and that the removal of a non-void odd or even wad from an eight-clique ring will render its complement uniquely four colorable. It seems most likely that analogues to Figures 4, 5, and 6 must exist for the four colorable case, and that corresponding theorems must be provable by corresponding techniques.

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